A Non-homothetic Multi-Sector Model of Business Cycles: Interaction of Skills, Consumption Patterns and Technology

Job Market Paper

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Abstract

In this paper, I study an economy with two types of agents (L and H-skill) and three consumption goods (basic goods, home and market provided service goods) as well as an investment good. Agents differ in terms of the jobs that they are specialized in and the composition of their consumption baskets. In particular, H-skill (L-skill, respectively) agents are specialized in the production of goods (services) and their consumption bundles contain relatively more goods (services) as well. Moreover, both types of agents have non-homothetic preferences which induce a strong desire to consume the good that the other type produces. In this environment, I show, through a series of quantitative experiments, that the imposed preference structure combined with the division of labor leads to a circular interaction between the two types of agents. This mechanism, in return, leads to an endogenous variation in the factor content of the goods consumed along the business cycle. I show that this proves to be very effective in generating highly persistent aggregate data in response to exogenous investment shocks and in amplifying magnitudes of the shocks.

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1 Introduction

In this paper, I consider an economy in which agents are specialized either in goods or service producing sectors and have different preferences over the goods produced in these two sectors. In this framework, specialization leads naturally to the set of goods which agents contribute to its production and their consumption bundles being different. This discrepancy between the consumption bundles and production activities in return introduces a new type of interdependency among the agents across the population, in addition to one that already present in general equilibrium models with homogeneous agents. Here, it arises as a result of the functional roles assumed through the division of labor in the economy. In this environment, I study how the demand for the type of labor an agent supplies and her consumption behavior interacts with each other in propagating the exogenous economic shocks.

In order to motivate the framework the model is set in and some of its assumptions, I will discuss, as an example, how work and leisure decisions of different agents are in fact related across the population. The type of interdependency that I mention above arises as soon as leisure activities\(^1\) are modeled as commodities rather than considering leisure as a fraction of the time endowment that agents can enjoy by doing nothing as modeled in the standard models. For example, according to the time-use surveys, the time allocated to watching T.V. is one of the main leisure activities (Aguiar and Hurst (2008)). But it is hard to argue that those who watch more T.V. derive more utility compared to those who do not watch much T.V. but instead utilize their time in different activities. Indeed, the same paper documents that in comparison to less educated men, more educated men spend less time on T.V., but more on other leisure activities like exercising, reading, and hobbies. Moreover, Aguiar and Hurst (2016), after showing that lower skilled individuals increased their leisure time between 1985 and 2003 while higher skilled individuals lowered their leisure time, they conclude: The increase in leisure inequality has matched the well-documented increase in income and consumption inequality during the last 30 years documented by many in the literature.\(^2\)

\(^{1}\)Leisure activities include time spent watching television, socializing, going to the movies, playing video games, exercising, and sleeping (see Aguiar and Hurst (2016)).
Figure 1 depicts the total weekly leisure choices by educational attainment and gender over the last three decades. The figure shows that both for men and women the total leisure time taken decreases as the level of education rises.\(^2\) It is hard to rationalize these patterns observed in the data if leisure is assumed to be a normal good just as any other consumption good. Otherwise, higher skilled individuals, who have had less time for **doing nothing** over the last 30 years, could simply imitate the lifestyle of lower skilled individuals. However, they do not and their lifestyle would be considered as more desirable.\(^3\) Therefore, I deemphasize the choice between work and leisure. Instead, I consider chosen consumption bundles and work decisions as pairs that are **jointly revealed** in the data as an outcome of some rational preferences over the set of feasible pairs which, I assume, are determined outside the model. In the previous T.V. example, I consider watching more T.V. as a revealed preference outcome given the feasible alternatives and usually it is more common among the lower skilled people for whom better options are **not feasible**.

In order to see how work-leisure decisions are interdependent in the model, let us look at what might happen after a positive technology shock: higher skilled individuals become more productive and they are inclined to buy more time-saving goods (like eating out or hiring a dog walker, etc.) to be able to work more in the saved time and buy higher intensity leisure activities (going to a Broadway show rather than going to the cinema or watching T.V. at home). Since other agents are specialized in the production of these time-saving jobs or higher intensity leisure activities, demand for their labor increases as well, even though the technology shock had no immediate impact on their productivity in the jobs that they are specialized in. In the model, this mechanism actively contributes to the propagation of the economic shocks.\(^4\)

Going back to the work-leisure choice discussion, in many cases, once an occupation/role is taken, there is not much flexibility to adjust the time spent at work. Think of someone who is about to decide between an academic job versus working in finance. These two professions offer two very different lifestyles, but the agent knows this at the time of decision and once

\(^2\)The figure also shows that the gap between the total leisure time taken by high and low skilled individuals has significantly increased over the last thirty years.

\(^3\)For lower skilled agents imitating the lifestyle of higher skilled individuals simply might not be feasible.

\(^4\)Also note that, this mechanism might be particularly important in capturing the movements in the labor market at the extensive margin over the business cycle, which has been considered to be more important (see Hansen (1985)) compared to adjustments at the intensive margin.
Figure 1: Leisure by Years of Education (YoE). Source: Comprised from Aguiar and Hurst (2007), Table V.
she chooses her profession she just fulfills her typical expected role in that lifestyle. Of course, in the same job, different people might end up with different amounts of time spent at work, but since we are interested in averages across consumers with similar characteristics these heterogeneities wash out within the group.

In summary, in this framework, the decision between working time and leisure is not as central as in the standard models. Instead, agents are modeled as if they chose a lifestyle given their skill endowments and then in response to the exogenous shocks, they adjust their lifestyle (consumption behavior in the model) as much as possible under the relevant economic constraints.

The model features a novel amplification mechanism that arises as a result of the combined effects of the differential factor contents of the goods consumed by different agents as well as the preference structure imposed. In particular, I assume that agents prioritize increasing the fraction of basic goods in their consumption bundles up to a certain level, and then the propensity to spend on market produced services increases, so the preferences are non-homothetic. The opposite is true for higher skilled individuals; their consumption bundles already contain a satisfactory level of goods so they expand their consumption towards services faster. Now since goods and services have different capital intensities, this preference structure gives rise to an endogenous variation in the average capital intensity of the goods consumed over the business cycle. Since the capital stock is the main driver and the only stock variable in the model, an endogenous variation in the average capital intensity of the goods consumed has first order effects on the statistical properties of the model. I show that under the maintained assumptions the model exhibits significant gains in terms of persistence and amplification of the shocks.

In the next sections, after a brief literature survey, first I present the model with complete specialization and then assess its performance through a series of quantitative experiments, and finally in the last section I extend the model to allow incomplete specialization.

2 Related Literature

In this section, I discuss how my model is related to several strands of the business cycle literature.
Home production models (Benhabib, Rogerson and Wright (1991)) are based on a related observation that there is a high degree of cyclicality in market consumption of goods and services that are substitutes for home production (such as eating at restaurants, housecleaning, child-care, etc). These models try to account for this observation by modeling the trade-off that agents face when they allocate their total time endowment between market and non-market activities over the business cycle, which in return, are driven primarily by changes in the market technology relative to the home technology. But since there is no specialization in these models, the circular interaction that I am after in this chapter does not appear in these models.

Chang (2000) and Bils, Chang and Kim (2012) proposes business cycle models with occupational choice based on comparative advantage. In contrast to these papers, in my model there are two types of agents with different consumption behaviors and the sectors differ in terms of their factor intensities.

Models of industrial input-output networks (Dupor (1999), Horvath (2000), Acemoglu et al. (2012)) focus on the production side of the economy and typically impose a network structure among the sectors and allow sectoral heterogeneity in terms of their capital intensities. In contrast to these papers, in my model the demand side of the economy receives more attention in the modeling and its interaction with the production structure in propagating the exogenous shocks is the central question I would like to investigate.

Similar to the business cycle models with investment specific shocks (Greenwood, Hercowitz and Huffman (1988), Greenwood, Hercowitz and Krusell (1997), Greenwood, Hercowitz and Krusell (2000)), I assume that exogenous shocks effect the marginal efficiency of investment but not existing capital and this generates a wedge in the user cost of capital across time, and firms respond to this wedge by adjusting their capacity utilizations.

In a recent paper, Jaimovich, Rebelo and Wong (2017) allow households to choose both the quantity and quality of the goods they consume and show that the choice of the quality of the goods consumed affects the total employment significantly. They assume that the labor intensity increases with the quality, i.e. the production of higher quality goods contain more labor. On the other hand, in my model, I emphasize the functional roles assumed in an economy based on the division of labor, which in return requires to allow a more flexible
relationship between labor intensity and quality. For example, in my model, in contrast to their assumption, market produced service goods are more labor intensive but they do not necessarily offer higher utility for all individuals.

Gali (1994) and Benhabib and Wen (2004) propose models with demand shocks and emphasize the compositional effects of demand on the dynamics of output as in my model. In more related work, Benhabib, Perli and Sakellaris (2006) discuss in detail how compositional effects may play a significant role in the dynamic properties of aggregate variables in the two, especially in the three-sector real business cycle models provided that the factor intensities among the sectors are different enough. However, in their model there is a single consumption good and their main focus is to produce the right output dynamics, which they define as hump-shaped impulse response functions and positive autocorrelation coefficients for at least a few periods after the shock.

Finally, this paper addresses some of the issues, most notably raised by Cogley and Nason (1995), related to performance of the real business cycle models. Cogley and Nason (1995) conclude that many RBC models have weak internal propagation mechanisms and do not generate interesting dynamics via their internal structure. As they mention in their paper, these models rely on two main propagation mechanisms: capital accumulation and intertemporal substitution. On the one hand, as the intertemporal substitution of labor-leisure and capital accumulation are flip coins of the same story in these models they can deliver an elegant story parsimoniously, but on the other hand, the same exact feature puts the modeler into a conceptual straitjacket. In this respect, the present model is just another attempt to build a business cycle model with a stronger internal propagation mechanism that arises as a result of the additional features mentioned above.

3 Preferences

3.1 A Discussion of Consumption Categories

As discussed in the introduction, instead of making the labor-leisure choice a central point in the modeling, I model agents as if they choose an optimal consumption bundle over

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5I skip the mechanisms that are based on adjustment costs or lags.
different goods as a function of their relative positions in the economy (their wage income in my model). Of course, the set of possible consumption goods that individuals choose from can be specified in many different ways but, I think in terms of its relevance to the business cycle research, the following two group categorization over goods and services\(^6\) is particularly useful in investigating interdependencies generated by the division of labor across the population.

(i) **Category of Goods** \((C_g)\): This category includes all the goods that do not require someone else’s labor service at the moment of consumption. However, they might be of various qualities. For simplicity of the exposition in this section, assume they come only in two versions; goods are either basic goods \((C_b)\) or customized goods \((C_c)\). The main difference between these two versions of goods is their skill intensities, the latter has a greater high-skill content.

(ii) **Category of Services** \((C_s)\): On the other hand, goods consumed under \(C_s\) should satisfy the following condition: they can potentially be met by the consumer themselves once the necessary inputs are purchased at the market, and if they are not met by the consumer, they require someone else’s labor service at the time of consumption. In the former case, the consumer produces the service at home but has to decide between the basic or customized inputs, \(C_b\) and \(C_c\), to produce the service goods at home \((C_h)\). On the other hand, in the latter case the consumer purchases the service from the market \((C_m)\), which is produced by firms using again \(C_b\) and \(C_c\) and labor, in addition. For instance, consider someone who would like to learn to play the guitar. If a computer software is purchased for that, the service is \(C_h\), and if a music teacher is hired the service consumed is \(C_m\).

In summary, consumers derive utility from goods \((C_g)\) and services \((C_s)\) according to a consumption aggregate

\[
C_t = C(C_{gt}, C_{st}),
\]

and in return both \(C_{gt}\) and \(C_{st}\) have two versions:

\[
C_{gt} \in \{C_{bt}, C_{ct}\}, \quad C_{st} \in \{C_{ht}, C_{mt}\},
\]

\(^6\)The distinction between goods and services is standard, but the way that I define these two categories is more specific.
that is, agents can consume either the basic version of the goods \((C_{bt})\) or the customized version \((C_{ct})\). Similarly, agents choose between market produced service goods, \(C_{mt}\), and home produced service goods, \(C_{ht}\).

In the following sections, I assume that only the basic good \((C_{bt})\) is produced from the goods category and the services category contain two versions as mentioned above.

### 3.2 Specification of Utility Functions

I assume that the preferences over goods and service categories are represented by the following Leontief function

\[
C_t = \min \{C_{st}, \xi C_{gt}^\nu\}
\]

Although this utility function implies that the elasticity of substitution between goods and services is zero, because of the nonlinearity in the second term, consumers expand their consumption bundles in favor of service goods as the consumption index (or real income) rises. The parameter \(\nu\) controls how fast consumers switch from goods to services as their real income rises.\(^7\) Therefore this particular specification of Leontief function leads to nonlinear income expansion (Engel) curves. More specifically, when \(\nu > 1\) this utility function implies that both goods are normal goods and service goods is a luxury good. In Figure 2 I plot the indifference curves and the income expansion path corresponding to \(\xi = 1\) and \(\nu = 2\). This view is consistent with the empirical results discussed in subsequent sections. Although this assumption might be extreme and partly made for the sake of simplicity, I think it is also a reasonable assumption for the following reasons. First, my main concern in this paper is to evaluate the contribution of endogenous variation in the average factor content of demand to the business cycle properties, not how endogenous variation in relative prices effect the allocation of resources between the consumption categories. Second, related to the first point, I do not have any theory for the variation of relative prices but empirically there is an ample evidence supporting the pattern depicted in Figure 2, so I consider this specification as an empirically supported discipline that will impose restrictions

\(^7\)Mathematically, \((\partial C_{st}/\partial C_{gt})(C_{gt}/C_{st}) = \nu\) therefore as the consumer moves across the indifference curves the composition of his consumption basket changes at a constant rate \(\nu\).
over how the other endogenous variables should behave in the model when the effect of relative prices is eliminated.\(^8\)

For the two goods, \(C_{ht}\) and \(C_{mt}\), consumed under service category, I assume that the preferences are implicitly defined by the following isoelastic specification

\[
1 = \sum_{j \in \{h,m\}} \Omega_j^\frac{1}{\sigma} C_{st}^{\epsilon_j - \sigma} C_{jt}^{\sigma - 1}. \tag{3.2.1}
\]

These preferences are non-homothetic generalizations of CES preferences.\(^9\) They were originally proposed in mid-seventies by Sato (1975) and Hanoch (1975) and they have recently been used in a few papers in the structural change literature (Comin, Lashkari and Mestieri (2015), Matsuyama (2017), Duernecker, Herrendorf and Valentinyi (2016), see also Appendix 1). The parameters in the utility function have the following interpretations: \(\sigma\) measures the elasticity of substitution between home produced services \((C_{ht})\) and market provided services \(C_{mt}\); \(\epsilon_j\) measures the income sensitivity of the share of expenditures on

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\(^8\)To clarify, Leontief functional form eliminates the relative price effect.

\(^9\)When \(\epsilon_h = \epsilon_m = 1\), these preferences reduce to the standard CES preferences

\[
C_{st} = \left( \sum_{j \in \{h,m\}} \Omega_j^\frac{1}{\sigma} C_{jt}^{\frac{\sigma - 1}{\sigma}} \right)^{-\frac{1}{\sigma - 1}}
\]
the good $C_{jt}$, $j \in \{h, m\}$. Through this non-homothetic CES specification, one can obtain independent income and substitution elasticities for market and home produced goods. In particular, Comin, Lashkari and Mestieri (2015) show that the demand functions resulting from these preferences have the following two distinctive properties:

1. The elasticity of the relative demand for home and market produced services with respect to the aggregate consumption index for the services is constant, i.e.,

$$\frac{\partial \log(C_{ht}/C_{mt})}{\partial \log C_{st}} = \epsilon_h - \epsilon_m.$$ 

2. The elasticity of substitution between home and market produced services is uniquely defined and constant

$$\frac{\partial \log(C_{ht}/C_{mt})}{\partial \log(P_{ht}/P_{mt})} = \sigma.$$ 

As they discuss, the first property ensures that the non-homothetic features of these preferences do not systematically vary as income grows, in contrast to, for example, Stone-Geary preferences. On the other hand, the second property ensures that the patterns of intra-sectoral substitution have a constant price elasticity and, thus, do not systematically vary as income grows. I find the latter property particularly useful since, as stated above, I have no theory for the evolution of relative prices over the business cycle, these preferences ensures that at least the demand side of the economy alone will not lead to the variations in the relative prices. I interpret this parameter $\sigma$ as the consumer’s willingness to diversify its consumption expenditures. As I will discuss in the next section, household spending becomes more diversified as income rises, which in return suggest that high income households should have a higher elasticity of substitution compared to lower income households ($\sigma_h > \sigma_l$) as will be the case in the quantitative exercises.

Finally, to represent the preferences over consumption and leisure I use preferences with no intertemporal wealth effect on the labor supply as in Greenwood, Hercowitz and Huffman (1988) (GHH henceforth).

$$U(C_{gt}, C_{ht}, C_{mt}N_{ht}) = \Phi_1 \left( C_t - \Phi_2 \frac{N_{ht}^{1+\theta}}{1+\theta} \right)^{1-\gamma} \quad (3.2.2)$$

11
where $C_t$ is implicitly defined by the following relationships

$$
C_t = \min \{C_{st}, \xi C_{st}^{\nu} \} \quad (3.2.3)
$$

$$
1 = \sum_{j \in \{h, m\}} I \sum_{j \in \{h, m\}} \Omega_{j} \frac{1}{\sigma} C_{st}^{\sigma-\epsilon} C_{st}^{\sigma-1}. \quad (3.2.4)
$$

In (3.2.3), the parameters $\xi$ and $\nu$ control the share of expenditure on service goods and the curvature of the income expansion path, respectively. On the other hand, in (3.2.4) the following parametric restrictions are sufficient to guarantee that the aggregator $C_{st}$ is globally monotonically increasing and quasi-concave, yielding a well-defined utility function over the bundle of goods $(C_{ht}, C_{mt})$: 1) $\sigma > 0$ and $\sigma \neq 1$, 2) $\Omega_{j} > 0$ for all $j$, and 3) $(\sigma - \epsilon_{i})/(\sigma - 1) > 0$ for all $j \in \{h, m\}$ (see Hanoch (1975), pp.403 and Comin, Lashkari and Mestieri (2015), pp. 6).

I assume that higher and lower skilled individuals have the same form of preferences as given in (3.2.2)-(3.2.4) but they differ in terms of the preference parameters:

$$
\Phi_{1i}, \Phi_{2i}, \theta_{i}, \xi_{i}, \nu_{i}, \sigma_{i}, \epsilon_{i}, \Omega_{i} \quad i \in \{h, \ell\}
$$

That is, the preferences of an agent depends on whether she is employed as a high skill or low skill worker. In the next section, I present some empirical evidence for this assumption and provide an extended discussion on its implications in the model.

### 3.3 Empirical Evidence on Cross-sectional Consumption Patterns

First support for the existence of different consumption patterns come from the survey paper by Chai, Rohde and Silber (2015) in which they list three stylized facts:

**Fact 1**: At low income levels, spending diversity is low as food expenditure dominates spending,

**Fact 2**: As household income grows, spending diversity increases via reductions in the budget share of food spending and increases in non-food expenditure

**Fact 3**: Individual household spending becomes more diversified as income rises.
Under the utility specification I employ in this paper, these stylized facts can be accounted for. In particular, nonhomothetic preferences can take care of the nonlinear expenditure share of food in the total consumption expenditures (Fact 1 and Fact 2), and the elasticity of substitution that varies over skill groups can approximate the more diversified spending pattern as income rises (Fact 3).

Next, I analyze the data published by the Bureau of Labor Statistics in their Consumer Expenditure Survey (CES). I classify each consumption category surveyed in CES into either luxury or necessity and look at how the total share of each category has evolved in the last 30 years. The results are presented in Figure 3. There are two important conclusions from this exercise:

i) There are stark differences among income quintiles in terms of the share of spending on luxuries out of the total spending. For instance, the highest income quintile spends approximately 65% on luxuries, whereas the lowest income quintile spends less than 40% of their total expenditures on luxuries.

ii) The share of luxuries in total expenditures decreases steadily from the highest income quintile to the lowest.

These two findings provide empirical support for the assumption that the preference parameters should vary across the income groups to reflect these different consumption patterns.\textsuperscript{10}

Therefore, based on all these empirical evidence, I conclude that the cross-sectional data clearly shows that the expenditure shares of goods and services vary across the population at a point in time: as the income level increases, consumption bundles contain more varieties and a higher fraction of services purchased at the market.

However, this cross-sectional variability of consumption bundles should not to be confused with preference heterogeneity because it is a preference pattern that arises mostly due to the income and wealth inequality. In order to develop some motivation for the assumption I make in this paper, let us use the same consumption aggregator $C_t \equiv C_{st}^g C_{st}^{1-\gamma}$.

Also in instead of assuming fixed expenditure shares of goods and services as implied by Cobb-Douglas preferences, suppose that there is a cross-sectional preference pattern such

\textsuperscript{10}In Appendix 4, I provide further evidence supporting this assumption based on Taylor and Houthakker (2009) and the detailed spending data presented in Household Spending: Who Spends How Much on What.
that agents approximately move along the same pattern as their income rises. For example, suppose that the share of expenditure on goods is given by the following time-invariant function

$$\gamma = \gamma(y),$$

where \( y \) is income of the agent. In other words, an agent with income \( y \) spends \( \gamma(y) \) fraction of her total consumption expenditures on goods, and \( 1 - \gamma(y) \) on services. This assumption also entails that agents just adopt the consumption bundles of higher income agents as their income increases.\(^{11}\) As an example, consider the following functional form

$$\gamma(y) = ae^{-by}, \quad a \in (0, 1), \quad y \in [0, \infty),$$

(3.3.1)

where the lowest income level is normalized to 0. According to this invariant population preference distribution, the expenditure share of goods is \( a \) for the lowest income agent, and \( ae^{-by} \) for the highest income agent as shown in the figure below. If the level of income of the highest type rises, say from \( y \) to \( y' \), then the expenditure share of goods decreases.

\(^{11}\)Alternatively, the population preference distribution can be modeled as a function of the agent’s rank in terms of their income.
from $ae^{-by}$ to $ae^{-by'}$. Similarly, other agents move towards the right depending on their new income levels.

![Figure 4: Cross-sectional Consumption Pattern](image)

In terms of general equilibrium thinking, this assumption should cause no trouble since in a business cycle model, the main concern should be to replicate the main aggregate statistics and eventually provide some predictions for the course of aggregate variables, not to track the evolution of the exact consumption bundles of agents over the business cycle. I claim that imposing such an empirically robust discipline in multi-sector business cycle models can only increase their applicability. Methodologically this paper can be considered as an attempt to incorporate a significant amount of heterogeneity into the model without rendering it intractable. Using a time-invariant preference structure simplifies the analysis in the following sense: since the agents simply move along a time-invariant curve, we only need to keep track of the measure of agents who push the current frontier of the consumption bundle further and the measure of agents who moved forward from the bottom part of the preference distribution. The difference between the measures of these two groups gives a good idea about the total change in the factor content of goods and services consumed as the economy transits from one point to another.
4 Model

4.1 Households’ Problem

I assume that there is a continuum of identical families of measure one and each family consists of a measure \( m \) of high-skill agents and a measure \( 1 - m \) of low-skill agents. I assume that head of the household acts like a social planner in each family and maximizes the total utility of the family members by choosing infinite sequences of consumption and labor supply for each type and managing the capital stock of the economy. Let us denote the total utility of the family members by

\[
U(\cdot) = U_h(C_{ht}, C_{ht}, C_{mt}, N_{ht}) + U_\ell(C_{\ell t}, C_{\ell t}, C_{\ell t}, N_{\ell t}).
\]

where the utility function of the high skill agents is the following\(^{12}\)

\[
U_h(C_{ht}, C_{ht}, N_{ht}) = \Phi_{1h} (C_{ht} - \Phi_{2h} \frac{N_{ht}^{1+\theta_h}}{1+\theta_h})^{1-\gamma}, \quad 0 < \gamma, 0 < \theta_h
\]

where \( C_{ht}, C_{ht} \) and \( C_{mt} \) are consumption of basic goods\(^\text{13}\), home produced and market provided services, respectively; \( N_{ht} \) denotes the labor supply of the agent; \( C_{ht} \) is the consumption aggregate that is implicitly defined through the following relations

\[
C_{ht} = \min \left\{ C_{ht}, \frac{C_{ht}}{h^\gamma} \right\}
\]

\[
1 = \sum_{j \in \{h, m\}} \Omega_{hj} C_{C_{ht}} C_{C_{ht}}^{\frac{\theta_j}{\theta_h} - 1}.
\]

Now, we can write the problem that the head of the households solves as follows

\[
V(K_t, q_t) = \max_{\{C_{gt}, C_{ht}, C_{mt}, N_{ht}, N_{lt}, I_t, u_t\}} \left\{ U(\cdot) + \beta \mathbb{E} V(K_{t+1}, q_{t+1}) \right\}
\]

\(^{12}\) Since the description of the problem is exactly the same for high and low skill agents I will stick to the problem of high skill agents to avoid additional notation.

\(^{13}\) I use the terms basic goods and goods interchangeably in the paper unless there is a risk of confusion.
subject to

\[(i) \sum_{j \in \{g,h,m\}} P_{jt}C_{jt} + I_t = R_t u_t K_t + w_{lt} N_{lt} + w_{ht} N_{ht}\]

\[(ii) K_{t+1} = (1 - \delta(u_t))K_t + I_t q_t,\]

\[(iii) C_{jt} = C_{hjt} + C_{\ell jt}, j \in \{g,h,m\},\]

\[(iv) C_{hgt}, C_{hst}, C_{\ell gt}, C_{\ell st} \geq 0,\]

\[(v) \text{given } K_0.\]

where \(P_{jt}\) is the price good \(j, j \in \{g,h,m\}\); \(I_t\) the total investment; \(R_t\) is the rental price of capital services; \(K_t\) is the aggregate capital stock; \(q_t\) is a stochastic term and \(w_{ht}\) (\(w_{lt}\)) is the wage rate for high-skill (low-skill) workers. In the model, all nominal variables are expressed in terms of the investment good.

In words, at the beginning of period \(t\) the family starts with \(K_t\) units of capital inherited from the previous period, and makes four decisions: (i) how much capital services to generate \((u_t K_t)\) in the current period by choosing the capacity utilization rate \(u_t\), (ii) how much new capital \((I_t)\) add to the current capital stock for a given efficiency level \(q_t\), (iii) how much labor to supply \((N_{lt} \text{ and } N_{ht})\), and (iv) how to allocate the total consumption expenditure between goods \((C_{gt} = C_{hgt} + C_{\ell gt})\) and services \((C_{st} = C_{hst} + C_{\ell st})\).

Also, as in GHH, the capital utilization decision involves Keynes’ notion of user cost. That is, a higher utilization rate causes a faster depreciation of the capital stock, either because wear and tear increase with the use or because less time can be devoted to maintenance. This effect is modeled as a variable depreciation cost \(\delta(u_t)\) in the capital accumulation equation above. The non-negative depreciation function \(\delta\) satisfies \(0 \leq \delta \leq 1, \delta' > 0, \delta'' > 0\). In particular, I assume \(\delta(u_t) = bu_t^\tau / \tau\). The variable \(I_t\) is a component of the gross investment, as corresponding to the national income accounts. Its contribution to the production capacity in \(t + 1\), however, depends on the technological shift factor \(q_t\), affecting the productivity of the new capital goods. The productivity of the already installed capital stock \(K_t\), is not directly affected by the new technology. Correspondingly, \(K_{t+1}\) is a measure of the future capital stock in productivity units.
4.2 Firms’ Problem

On the production side, there are three representative firms producing basic goods ($Y_{gt}$), market services ($Y_{mt}$) and investment goods ($Y_{It}$), respectively. Each firm solves the usual static problems so I outline their problems only briefly below.

**Basic Goods Production.** The representative firm that produces (basic) goods, $Y_{gt}$, employs high-skill labor ($H_t$) and capital ($K_{gt}$) in production, and solves:

$$\max_{K_{gt}, H_t} P_{gt} A_{gt} K_{gt}^{1-\alpha_g} H_t^{\alpha_g} - R_t K_{gt} - w_{ht} H_t,$$

where $K_{gt}$ is the capital services rented by the goods producing firm.

**Market Produced Service Goods.** Only low-skill labor and capital are required to produce market services$^{14} C_m$, so the representative firm that produces service goods in the

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$^{14}$This approximately corresponds to *service occupations*. For example, babysitting, dining out or recreational services can be considered as examples of $C_m$. However, in principle $C_m$ might include some other input combinations as well. For example, any kind of non-compulsory educational service should be considered as a market service in this categorization, and be counted under $C_m$. 

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market, $Y_{st}$, solves:

$$
\max_{K_{st}, L_t} P_{st} A_{st} K_{st}^{1-\alpha_s} L_t^{\alpha_s} - R_t K_{st} - w_t L_t.
$$

**Home Produced Service Goods.** I assume that there is a linear technology producing services at home $C_h$ from basic goods: $C_h = \eta_j C_b$, where $\eta_j$ is the productivity of type $j \in \{h, \ell\}$ agents in transforming basic goods into service goods.

**Investment Goods Production.** There is a separate investment good producing firm which uses inputs from both basic goods ($M_{gt}$) and market services ($M_{st}$) to produce the investment good according to a constant elasticity of substitution production function

$$
Y_{It} = A_{It} \left( \omega_g \left( M_{gt}^{\frac{-\epsilon}{1-\epsilon}} + \omega_s M_{st}^{\frac{-\epsilon}{1-\epsilon}} \right)^{\frac{1}{1-\epsilon}} \right)
$$

where $\epsilon > 0$ is the elasticity of substitution across input categories and $\omega_k, k \in \{g, s\}$, is the relative weight of each input ($\omega_g + \omega_s = 1$).

This specification allows for different degrees of complementarity: if $\epsilon < 1$, inputs are gross complements, and if $\epsilon > 1$, inputs are gross substitutes. In the two extreme cases: $Y_{It}$ is Cobb-Douglas, $Y_{It} = M_{gt}^{\omega_g} M_{st}^{\omega_s}$, for $\epsilon = 1$, and $Y_{It}$ is Leontief, $Y_{It} = A_{It} \min\{M_{gt}/\omega_g, M_{st}/\omega_s\}$, for $\epsilon = 0$.

The representative firm producing the investment good solves:

$$
\max_{M_{gt}, M_{st}} A_{It} \left( \omega_g M_{gt}^{\frac{-\epsilon}{1-\epsilon}} + \omega_s M_{st}^{\frac{-\epsilon}{1-\epsilon}} \right)^{\frac{1}{1-\epsilon}} - P_{gt} M_{gt} - P_{st} M_{st}
$$

and the first order conditions of this problem determines the demand for each good by the investment firm:

$$
M_{gt} = \frac{\omega_g P_{gt}^{1-\epsilon}}{\omega_g P_{gt}^{1-\epsilon} + \omega_s P_{st}^{1-\epsilon}} Y_{It}
$$

$$
M_{st} = \frac{\omega_s P_{st}^{1-\epsilon}}{\omega_g P_{gt}^{1-\epsilon} + \omega_s P_{st}^{1-\epsilon}} Y_{It}
$$

Substituting the expressions for $M_{gt}$ and $M_{st}$ into $Y_{It}$ gives

$$
1 = A_{It}^{\frac{-\epsilon}{1-\epsilon}} \left[ \omega_g P_{gt}^{1-\epsilon} + \omega_s P_{st}^{1-\epsilon} \right]
$$
which in return allows us to rewrite the factor demand functions simply as

\[ M_{gt} = A_{ft}^{\epsilon - 1} \omega_g p_{gt} Y_{ft} \]
\[ M_{st} = A_{ft}^{\epsilon - 1} \omega_s p_{st} Y_{ft}. \]

4.3 Equilibrium

An equilibrium for this economy can be defined as follows: households and firms solve their respective problems as stated in the previous sections and market clearing conditions hold in each market:

- The market clearing conditions for basic goods and services

\[ Y_{gt} = M_{gt} + C_{hgt} + C_{tgt} + \eta_h C_{hht} + \eta_l C_{lht} \]
\[ Y_{st} = M_{st} + C_{hmt} + C_{lmt}. \]

- Market clearing conditions for labor:

\[ H_t = N_{ht} \]
\[ L_t = N_{lt}. \]

- Markets clearing conditions for capital and investment:

\[ u_t K_t = K_{gt} + K_{st} \]
\[ I_t = Y_{ft} \]

where \( u_t K_t \) is the total capital services supplied by the household.

Finally, I define the GDP as the total gross value of consumption and investment expressed in terms of the investment good:

\[ Y_t = C_t + I_t, \]
where $C_t = \sum_{j \in \{g,h,m\}} P_{jt} C_{jt}$ is the value of total consumption expressed in terms of the investment good.
5 A Quantitative Assessment of the Model

In this section, I first characterize the steady state of the economy and then run a series of experiments to assess the performance of the model under different scenarios.

5.1 Steady State

In Appendix 2, I show that the following set of equations characterizes equilibrium of the economy:

\[-\Phi_{1h}U_{ht}^{-\gamma} + \xi_t \frac{1}{\nu} \mu_{1t} C_{ht}^{\frac{1}{\nu} - 1} = \eta_h \left( \frac{\epsilon_{th} - \sigma_h}{\sigma_h - 1} \right) \left( \frac{C_{htt}}{C_{ht}} \right) \mu_{1t} + \left( \frac{\epsilon_{ht} - \sigma_h}{\sigma_h - 1} \right) \left( \frac{C_{ht}}{C_{ht}} \right) \mu_{2t} \]

\[-\Phi_{1t}U_{tt}^{-\gamma} + \xi_t \frac{1}{\nu} \mu_{1t} C_{tt}^{\frac{1}{\nu} - 1} = \eta_t \left( \frac{\epsilon_{tt} - \sigma_t}{\sigma_t - 1} \right) \left( \frac{C_{tt}}{C_{tt}} \right) \mu_{1t} + \left( \frac{\epsilon_{tt} - \sigma_t}{\sigma_t - 1} \right) \left( \frac{C_{tt}}{C_{tt}} \right) \mu_{2t} \]

\[\Phi_{1h} \Phi_{2h} U_{ht}^{-\gamma} N_{ht}^{1+\theta_h} = \mu_{1t} \alpha_g Y_{gt} \]

\[\Phi_{1t} \Phi_{2t} U_{tt}^{-\gamma} N_{tt}^{1+\theta_t} = \mu_{2t} \alpha_s Y_{st} \]

\[\mu_{1t} \mu_{2t} \left[ \begin{array}{c} M_{st} \frac{1}{\omega_s I_t} \frac{1 - \epsilon}{A_{t\epsilon}^{1-\epsilon}} \left( \frac{Y_{1t} K_{1t}^{-1}}{u_t^{\gamma-1}} \right) = \left( 1 - \alpha_s \right) Y_{st} \end{array} \right] \]

\[Y_{gt} = \left( M_{gt} + \xi_t \frac{1}{\nu} C_{ht}^{\frac{1}{\nu}} + \eta_h C_{hht} + \xi_t \frac{1}{\nu} C_{tt}^{\frac{1}{\nu}} + \eta_t C_{ttt} \right) \left( 1 + \left( 1 - \frac{1}{\tau} \right) u_t^{\gamma} \right) \]

\[Y_{st} = M_{st} + C_{ht} + C_{ttt} \]

\[I_t = \frac{K_{t+1}}{q_t} - \frac{1 - \delta(u_t)}{q_t} \]
\[ u_t K_t = K_{gt} + K_{st} \]
\[ Y_{gt} = A_{gt} K_{gt}^{1-\alpha_g} N_{ht}^{\alpha_g} \]
\[ Y_{st} = A_{st} K_{st}^{1-\alpha_s} N_{\ell t}^{\alpha_s} \]
\[ I_t = A_{It} \left( \frac{1}{\omega_g} M_{gt}^{\frac{\epsilon_g}{1-\epsilon_g}} + \frac{1}{\omega_s} M_{st}^{\frac{\epsilon_s}{1-\epsilon_s}} \right) \]
\[ 1 = \sum_{i \in \{g, s\}} \Omega_{hi}^{\epsilon_h \sigma_h} C_{hi}^{\epsilon_h \sigma_h} C_{hit}^{\sigma_h \sigma_h \epsilon_h \sigma_h} \]
\[ 1 = \sum_{i \in \{g, s\}} \Omega_{\ell i}^{\epsilon_\ell \sigma_\ell} C_{\ell i}^{\epsilon_\ell \sigma_\ell} C_{\ell it}^{\sigma_\ell \sigma_\ell \epsilon_\ell \sigma_\ell} \]

In the system, there are 19 equations in 19 variables:

\[ C_h, C_{hh}, C_{hm}, C_\ell, C_{\ell h}, C_{\ell m}, N_h, N_\ell, u, M_g, M_s, K_g, K_s, K, Y_g, Y_s, I, \mu_1, \mu_2 \]

where, for convenience, I also define \( U_{jt} \equiv C_{jt} - (1/\theta_j) N^{1+\theta_j} \), \( j \in \{h, \ell\} \).

### 5.2 Experiments

Since my model extends the standard RBC model in a number of dimensions, I try to assess the contribution of these aspects by running a series different scenarios in this section.

First, I assume that the investment shock \( q_t \) follows an AR(1) process

\[ q_t = \rho q_{t-1} + \nu_t, \quad \nu_t \sim N(0, \sigma_f^2). \]

In order to choose the parameter values of the stochastic process I follow the same procedure as in GHH. Accordingly I set the magnitude of the innovation and the persistence of the stochastic process as \( \sigma_f = 0.045 \) and \( \rho = 0.52 \), respectfully. I also borrow the parameter values for \( \tau, \theta_h, \sigma \) and \( \beta \) from GHH, and set the fraction of high skill agents to one half, throughout the paper. Finally, I assume that the basic goods constitute 70% of the investment good in the steady state and the elasticity of substitution between basic goods and services is 0.1. I report other parameter values together with the results for each scenario below.
Scenario 1: Homothetic Preferences and Equal Capital Intensities \( \approx \) GHH

In order to understand the role of non-homothetic preferences in a multi-sector model I start with a scenario that is approximately equal to a standard one sector model with GHH preferences. For this scenario I assume that the capital intensities across the sectors are the same \((\alpha_g = \alpha_s)\) and agents have homothetic preferences with the same preference parameters. As an example of this scenario, I choose the set of parameter values as in Table 1.

| \(\gamma\) | 2.00 | \(\theta_h\) | 0.60 | \(\theta_\ell\) | 0.60 | \(C/Y\) | 0.80 |
| \(\beta\)  | 0.96 | \(\Phi_{h1}\) | 1.00 | \(\Phi_{\ell1}\) | 1.00 | \(I/Y\) | 0.20 |
| \(A_g\)   | 2.50 | \(\Phi_{h2}\) | 1.00 | \(\Phi_{\ell2}\) | 1.00 | \(K/Y\) | 2.06 |
| \(A_s\)   | 2.50 | \(\Omega_h\)  | 0.50 | \(\Omega_\ell\)  | 0.50 | \(K_g/uK\) | 0.76 |
| \(\alpha_g\) | 0.71 | \(\sigma_h\)  | 1.40 | \(\sigma_\ell\)  | 1.40 | \(Y_g/Y\) | 0.76 |
| \(\alpha_s\) | 0.71 | \(\epsilon_{hh}\) | 1.00 | \(\epsilon_{th}\) | 1.00 | \(P_sY_s/Y\) | 0.24 |
| \(\tau\)  | 1.42 | \(\epsilon_{hm}\) | 1.00 | \(\epsilon_{\ell m}\) | 1.00 | \(U_h/U_\ell\) | 1.00 |
| \(\omega\) | 0.70 | \(\xi_h\)  | 1.00 | \(\xi_\ell\)  | 1.00 | \(\rho\) | 0.51 |
| \(\epsilon\) | 0.10 | \(\nu_h\)  | 1.00 | \(\nu_\ell\)  | 1.00 | \(\sigma\) | 0.045 |

Table 1: Scenario 1- Paramater Values and Ratios

Table 2 presents the moments of the model for the parameter values given in Table 1. For convenience I also report the corresponding table from GHH together with the US data as reported in their paper. As Table 2 clearly shows, when the capital intensities are the same and the preferences are homothetic, moments of this model is almost identical to those from the model in GHH.\(^{15}\) Therefore, I take these parameter values as my reference point when I include non-homotheticities and differential capital intensities across the sectors in the next section.

Scenario 2: Non-homothetic Preferences and Equal Capital Intensities

Now I turn to a more realistic case as my second scenario. The following interpretation is useful to keep in mind as the main motivation behind this scenario. First I explain the intuition behind the parametric restrictions on the parameters that represent budget shares of different goods, and then on the parametric restrictions to represent marginal tendencies to buy different goods as income of consumers rises.

\(^{15}\)Impulse response functions of these two models are also very similar, see section 7
Table 2: Scenario 1- Moments

In the model, basic goods represent more sophisticated goods and therefore they are produced by high skill agents, and intuitively consumption baskets of high skill agents should contain more of these goods as well. Furthermore, I assume that consumption baskets of high skill agents contain more market produced services relative to consumption baskets of low skill agents. With the particular utility form discussed in subsection 3.2 these assumptions translates into the parametric restrictions $\xi_h < \xi_\ell$ and $\Omega_h < \Omega_\ell$.

As I discussed in section 1, I assume that agents prioritize increasing the fraction of basic goods in their consumption bundles up to a certain level, and then the propensity to spend on services increases, so the preferences are non-homothetic. Related with the discussion above, I assume that consumption baskets of high-income groups already contain satisfactory amount of basic goods, so they are inclined to buy more service goods ($\nu_h > 1$) and between home produced and market produced service goods they are inclined to buy more market produced service goods ($\epsilon_{hh} < 1 < \epsilon_{hm}$), like time-saving jobs supplied by low skill agents. On the other hand, lower skilled agents prioritize increasing the fraction of basic goods and home produced service goods in their consumption baskets ($\nu_\ell \leq 1$ and $\epsilon_{\ell m} \leq 1 \leq \epsilon_\ell h$).

This scenario, for example, can be represented by the set of parameter values given in Table 3.

Comparing the results in Table 2 and Table 4 we see that the standard deviation of output rises to 5.35 from 3.52, and its persistence rises to 0.70 from 0.66 after non-homothetic preferences are introduced. A standard criticism of GHH model is that they lead to unreal-
Table 3: Scenario 2- Parameter Values and Ratios

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>2.00</td>
<td>Output 3.50</td>
</tr>
<tr>
<td>$\theta_h$</td>
<td>0.60</td>
<td>USA Std 3.50</td>
</tr>
<tr>
<td>$\theta_\ell$</td>
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<td>USA AR 2.00</td>
</tr>
<tr>
<td>$C/Y$</td>
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<td>USA Corr 1.00</td>
</tr>
<tr>
<td>$\beta$</td>
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<td>GHH Std 3.50</td>
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<tr>
<td>$\Phi_{h1}$</td>
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<td>GHH AR 2.00</td>
</tr>
<tr>
<td>$I/Y$</td>
<td>0.20</td>
<td>GHH Corr 1.00</td>
</tr>
<tr>
<td>$\Phi_{\ell1}$</td>
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<td>Scenario 2 Std 3.50</td>
</tr>
<tr>
<td>$A_g$</td>
<td>2.50</td>
<td>Scenario 2 AR 2.00</td>
</tr>
<tr>
<td>$\Phi_{h2}$</td>
<td>1.00</td>
<td>Scenario 2 Corr 1.00</td>
</tr>
<tr>
<td>$K/Y$</td>
<td>2.06</td>
<td></td>
</tr>
<tr>
<td>$A_s$</td>
<td>2.50</td>
<td></td>
</tr>
<tr>
<td>$\Omega_h$</td>
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<td></td>
</tr>
<tr>
<td>$K_g/uK$</td>
<td>0.62</td>
<td></td>
</tr>
<tr>
<td>$\Omega_\ell$</td>
<td>0.70</td>
<td></td>
</tr>
<tr>
<td>$\epsilon_{h1}$</td>
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<tr>
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<td>$\sigma_\ell$</td>
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<td>$\alpha_g$</td>
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<td>$\sigma_h$</td>
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<td>$\alpha_s$</td>
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<tr>
<td>$\epsilon_{h2}$</td>
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<tr>
<td>$\epsilon_{\ell2}$</td>
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<tr>
<td>$\tau$</td>
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<td>$\epsilon_{h2}$</td>
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<td>$\epsilon_{\ell2}$</td>
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<tr>
<td>$\omega$</td>
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<tr>
<td>$\xi_h$</td>
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<tr>
<td>$\rho$</td>
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<tr>
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<td>$\nu_h$</td>
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<tr>
<td>$\nu_\ell$</td>
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<td></td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.045</td>
<td></td>
</tr>
</tbody>
</table>

Table 4: Scenario 2- Moments

The impulse response functions (IRFs) given in Figure 6 offer a better opportunity to understand the underlying reasons that lead to amplification in the model. In the figure, solid and dashed lines denote IRFs with homothetic (Scenario 1) and non-homothetic preferences (Scenario 2), respectively. The figure suggests the following interpretation: low skill agents would like to consume more basic goods and this is possible because high skill agents feel exactly the same for the service goods, therefore the effect of non-homotheticities on prices cancel each other out and consequently agents are able to increase their immediate consumption. In the figure, this can be seen from how IRFs jump up and start with positive intercepts after non-homotheticities are introduced. Recall that in standard models, to which our two sector model reduces with symmetric preferences, equal factor intensities and symmetric volatility in investment. However, as a notable feature of my model, this problem does not arise here; the volatility of investment remain almost the same.\textsuperscript{16}

Volatility of other variables remain approximately the same as well, except the volatility of consumption that I will discuss in the next section.

\textsuperscript{16}
homothetic preferences as assumed in this scenario, agents prefer to increase their savings to take the advantage of higher interest rates now and start consuming more a little later (dashed lines in the figure corresponds approximately IRFs of a single sector model). But here agents would like to expand their consumptions so badly towards the good that they do not produce, the motive to smooth consumption intertemporally becomes less important in comparison to the opportunity to increase their consumption in the desired direction immediately. But part of this immediate consumption would otherwise turn into the investment good. As a result, the total investment declines and a positive investment shock becomes less effective in boosting the economy. Therefore, the presence of non-homotheticity effectively works as if agents have a lower elasticity of intertemporal substitution (EIS) in the model. Consequently, agents invest less and consume more in response to a positive shock, and the model generates more amplification and persistence in investment. Also note that when two types of agents have different non-homotheticities, the model effectively works as if there are two types of agents with different EISs. This future has been proven to have important implications for the real business cycle models. For example, Guvenen (2006) shows that an otherwise standard real business cycle model featuring two classes of agents with different EIS is able to produce findings consistent with both capital and consumption fluctuations, and these findings can be reconciled with a low aggregate EIS as long as most of the wealth is held by a small fraction of the population with a high EIS. In these models, EIS of an agent typically depends whether the agent is a saver or borrower. My model provides another mechanism to have differential EIS in business cycle models by showing how non-homothetic preferences and EIS are related.

Scenario 3: Non-homothetic Preferences and Different Capital Intensities

In this scenario, I allow goods and service sectors to operate with different capital intensities. In particular, I assume that high skill agents work in a relatively more capital intensive sector, \( \alpha_g < \alpha_s \). This implies that the gross complementarity between capital and labor is higher for high skill jobs.\(^{17}\) To allow the total hours supplied by low skill agents to vary

\(^{17}\) However, note that the opposite of what I assume in this scenario (\( \alpha_g > \alpha_s \)) can also be justified under the following interpretation: ideas or frontier technologies are the ultimate form of scarcity in the economy and they are reflected in the high skill agents’ labor relatively more. As new methods are invented to implement the frontier technologies more cost effectively, the dependence for the high skill labor decreases...
more as in the data, I also assume that the opportunity cost of working is higher for high skill workers, $\theta_\ell < \theta_h$.

\[
\begin{array}{cccccccc}
\gamma & 2.00 & \theta_h & 0.60 & \theta_\ell & 0.30 & C/Y & 0.80 \\
\beta & 0.96 & \Phi_{h1} & 1.00 & \Phi_{\ell1} & 1.00 & I/Y & 0.20 \\
A_g & 2.50 & \Phi_{h2} & 1.00 & \Phi_{\ell2} & 1.00 & K/Y & 2.06 \\
A_s & 2.50 & \Omega_h & 0.40 & \Omega_\ell & 0.70 & K_g/uK & 0.62 \\
\alpha_g & 0.66 & \sigma_h & 1.40 & \sigma_\ell & 1.40 & Y_g/Y & 0.62 \\
\alpha_s & 0.80 & \epsilon_{h\ell} & 0.90 & \epsilon_{h\ell} & 1.00 & P_sY_s/Y & 0.38 \\
\tau & 1.42 & \epsilon_{hm} & 1.20 & \epsilon_{\ell m} & 1.00 & \mathcal{U}_h/\mathcal{U}_\ell & 1.81 \\
\omega & 0.70 & \xi_h & 0.70 & \xi_\ell & 1.00 & \rho & 0.51 \\
\epsilon & 0.10 & \nu_h & 2.00 & \nu_\ell & 1.00 & \sigma & 0.045 \\
\end{array}
\]

Table 5: Scenario 3- Parameter Values and Ratios

Comparing the results we see that the standard deviation of output rises to 7.16 from 5.35, and its persistence rises to 0.72 from 0.70 after differential capital intensities and labor elasticities are allowed.

and they can eventually be performed by relatively lower skilled agents as well. Therefore, capital can be considered as the part of the frontier technology that have been automized by these new methods.
Table 6: Scenario 3- Moments

The mechanics of the model again can be better understood from the sectoral IRFs that are depicted in Figure 7. From the graphs, it is clear that agents increase their consumption very sharply for both goods (compare the solid and the dashed lines), and what makes this possible is the direction that agents want to increase their consumption: each type has a very strong desire to consume the good that the other type produces. In other words, with the particular form of non-homothetic preferences that are assumed in this scenario, agents reciprocate to each other’s preferences, which in return leads to a circular interaction between them.

In this scenario, it is essential that both agents have non-homothetic preferences to allow the type of the circular interaction mentioned above to take place.

6 Extension: Incomplete Specialization

The model presented in section 4 assumes that agents are completely specialized in the production of a specific good. In this section I discuss, as a natural extension, how the results change when the specialization is incomplete and therefore agents can switch back and forth between the sectors over the business cycle. I assume that each household $s \in [0, 1]$ is endowed with a vector of efficiency units as a low and high skill worker, denoted by $e_l(s)$ and $e_h(s)$, respectively. Workers have the same efficiency units (normalized to unity) at performing low skill jobs, i.e. $e_l(s) = 1, \forall s$. However, agents have heterogeneous

---

18 The extension presented in section closely follows the model in my paper An Aggregative Model of Business Cycles.
skills in performing high skill jobs. Since this is the only heterogeneity among the agents, I will use the same letter to denote the agent’s efficiency units in goods production, i.e. $s$ equal the worker’s skill in high skill jobs, measured in efficiency units. I assume that skills $s$ are distributed according to a density function $f(s)$ across the population.

For simplicity, I assume that efficiency units (skills) are distributed exponentially on the interval $[0, \infty]$ with the density function $f(s) = \lambda e^{-\lambda s}$. The optimal allocation of labor between two sectors requires that there is a cut-off skill level $s_t$ such that agents with skill below $s_t$ take low skill jobs and the rest work as high skill workers (see Figure 8).

The planner’s problem with incomplete specialization is solved in Appendix 4.

7 Conclusions

In this paper, I studied an economy with two types of agents, heterogenous and non-homothetic preferences and different degrees of specialization. In this environment, through a series of numerical exercises I show that the division of labor together with non-homothetic
preferences are highly effective in propagating and amplifying exogenous shocks. I also discuss that the type of the economic environment proposed in this chapter offer an alternative framework to incorporate different features into the business cycles models. One example of this type that I mention in this chapter is the ability of the model to reconcile two different EISs as in saver-borrower models, but with plausible labor market movements.

Although I do not discuss in this chapter, the model has no difficulty in generating macroeconomic data that move all together over the business cycle. In particular, the impulse response functions of output, consumption and investment show that what is known as comovement puzzle in the literature does not arise in this framework.

Another issue that I do not address in this paper is whether the model can produce the right output dynamics, which Benhabib, Perli and Sakellaris (2006) define as hump-shaped impulse response functions and positive autocorrelation coefficients for at least a few periods after the shock. As pointed out by Benhabib, Perli and Sakellaris (2006), the post-impact inverse relationship between consumption and labor of a non-permanent shock is the main reason why the standard RBC models do not produce the right output dynamics. Although my model does not generate hump-shaped impulse response functions for the parameter values chosen in this chapter, this inverse relationship between consumption and labor does not appear in the model, which suggests that including some sorts of labor or capital adjustment costs as in Cogley and Nason (1995) might be an interesting extension to see
whether the model can generate hump-shaped output dynamics under these modifications.
References


Appendix 1

Implicitly Defined Utility Functions

The utility function \( v = f(x) \) is implicitly additive, if it may be defined by an identity of the form:

\[
\sum_k F^k(x_k, v) \equiv 1 \tag{7.0.1}
\]

where \( F^k \) are \( n \) functions of two variables (Hanoch (1975)).

Solving Implicitly Defined Problems

As Hanoch (1975) notes, the problem can be solved either by cost minimizing at given \( v \), or for utility maximization under budget \( c \) and prices \( p \). I provide some details below how to proceed in each case.

**Expenditure (cost) minimization:**

\[
\min \sum_k p_k x_k \quad \text{s.t.} \quad \sum_k F^k(x_k, v) = 1
\]

with FOCs:

\[
\lambda F^i_i = p_i, \quad \forall i
\]

Here \( \lambda \) is a Lagrange multiplier, a function of the price vector \( p \) and \( v \); it is not marginal cost, since \( F^i_i \neq \partial f / \partial x_i = -F^i_i / \sum_k F^k_v \). (Hanock 75, footnote 17)

**Utility maximization:**

\[
\max v \quad \text{s.t.} \quad \sum_k p_k x_k = c, \quad \sum_k F^k(x_k, v) = 1
\]

Set up the Lagrangian\(^{20}\)

\[
\mathcal{L} = v + \rho_1 \left( c - \sum_k p_k x_k \right) + \rho_2 \left( 1 - \sum_k F^k(x_k, v) \right)
\]

FOCs

\[
1 = \rho_2 \sum_k F^k_v
\]

\[
\rho_1 p_i = -\rho_2 F^i_i
\]

\(^{19}\)It is assumed that \( F^k \) are continuously twice differentiable, and satisfy other requirements for a unique, monotone, quasi-concave and closed \( f(x) \) for all \( x >> 0 \) (i.e. for all \( x_i > 0 \)). We rule out in this discussion the linear-isoquant function \( \sum f^k(v) \cdot v = 1 \), so that \( F_{kk} \neq 0 \). In principle, the right hand side of (7.0.1) may be a function \( h(v) \), but no loss of generality occurs if \( h(v) \) is incorporated in \( F^k \).

\(^{20}\)In Lagrangian formulation we consider \( F(x_1, \ldots, x_n, v) = v \) as the objective function and the constraints as two functions \( g_1 \) and \( g_1 \) that are again functions of the same \( n + 1 \) variables.
This can be expressed as a single condition

\[ p_i = \tilde{\lambda} F_i, \]

where

\[ \tilde{\lambda} = -\frac{1}{\rho_1 \sum_k F^k_v} \]

In Comin 2015, they set up the problem as an utility maximization but they don’t work out the first order condition with respect to \( v \) because it turns out that because of the specific \( F^k \) they choose they only need the ratio of Lagrange multipliers when the second condition is aggregated over the goods.

Also setting up the problem in two different ways clarifies why \( \lambda \) is a function of \( p \) and \( v \), as noted by Hanock 75.
Appendix 2

The planner’s problem with complete specialization is to find a pair of welfare weights \( \{ \Phi_{1\ell}, \Phi_{1h} \} \) and a sequence of contingency plans \( \{ C_{jgt}, C_{jht}, C_{jmt}, N_{jt} \} \) for type-\( j \in \{ \ell, h \} \) consumers and for aggregate allocations \( \{ K_t, K_{gt}, K_{st}, I_t, M_{gt}, M_{st}, u_t \} \) such that the sum of individual utilities that are weighted by the corresponding welfare weights is maximized:

\[
\max_{\{ C_{it}, C_{ijt}, N_{it}, M_{jt}, K_{jt}, K_{jt+1}, u_t \} \in \{ t, h \}, j \in \{ g, h, m \}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ U_h(C_{ht}, N_{ht}) + U_\ell(C_{\ell t}, N_{\ell t}) \\
+ \mu_{1,t} \left[ A_{gt} K_{gt}^{1-\alpha_g} H_t^{\alpha_g} - C_{gt} - M_{gt} \right] \\
+ \mu_{2,t} \left[ A_{st} K_{st}^{1-\alpha_s} L_t^{\alpha_s} - C_{st} - M_{st} \right] \\
+ \mu_{3,t} \left[ A_{lt} \left( \frac{1}{\omega_g^s} M_{gt}^{\frac{\alpha_g}{1-\alpha_g}} + \frac{1}{\omega_s^s} M_{st}^{\frac{\alpha_s}{1-\alpha_s}} \right)^{\frac{1}{1-\alpha_t}} - I_t \right] \\
+ \mu_{4,t} \left[ K_{t+1} - (1 - \delta(u_t)) K_t - I_t q_t \right] \\
+ \mu_{5,t} \left[ u K_t - K_{gt} - K_{st} \right] \\
+ \mu_{6,t} \left[ C_{gt} - C_{hgt} - C_{\ell gt} - \eta_h C_{hht} - \eta_\ell C_{\ell ht} \right] \\
+ \mu_{7,t} \left[ C_{st} - C_{hmt} - C_{\ell mt} \right] \\
+ \mu_{8,t} \left[ H_t - N_{ht} \right] \\
+ \mu_{9,t} \left[ L_t - N_{\ell t} \right] \\
+ \mu_{10,t} \left[ \sum_{j \in \{ h, m \}} \Omega_j^{\sigma_h} \frac{C_{hjt}^{\sigma_h - \sigma_h}}{C_{hjt}^{\sigma_h}} C_{hjt}^{\frac{\sigma_h - 1}{1-\sigma_h}} - 1 \right] \\
+ \mu_{11,t} \left[ \sum_{j \in \{ h, m \}} \Omega_j^{\sigma_\ell} \frac{C_{\ell jt}^{\sigma_\ell - \sigma_\ell}}{C_{\ell jt}^{\sigma_\ell}} C_{\ell jt}^{\frac{\sigma_\ell - 1}{1-\sigma_\ell}} - 1 \right] \right\}
\]

where

\[
U_h(C_{hgt}, C_{hht}, C_{hmt} N_{ht}) = \frac{\Phi_{1h}}{1 - \gamma} \left( C_{ht} - \Phi_{2h} N_{ht}^{1 + \theta_h} \right)^{1-\gamma}
\]

\[
U_\ell(C_{\ell gt}, C_{\ell ht}, C_{\ell mt} N_{\ell t}) = \frac{\Phi_{1\ell}}{1 - \gamma} \left( C_{\ell t} - \Phi_{2\ell} N_{\ell t}^{1 + \theta_\ell} \right)^{1-\gamma}
\]
and \( C_h \) and \( C_\ell \) are implicitly defined by the following relationships

\[
C_{ht} = \min \left\{ C_{hst}, \xi_h C_{hgt}^{\nu_h} \right\}
\]

\[
C_{\ell t} = \min \left\{ C_{\ell st}, \xi_\ell C_{\ell gt}^{\nu_\ell} \right\}
\]

\[
1 = \sum_{j \in \{h,m\}} \Omega_{hj}^{\frac{1}{\alpha_h}} C_{hst}^{\frac{1}{\alpha_h}} C_{hjt}^{\frac{\sigma_h-1}{\alpha_h}}
\]

\[
1 = \sum_{j \in \{h,m\}} \Omega_{\ell j}^{\frac{1}{\alpha_\ell}} C_{\ell st}^{\frac{1}{\alpha_\ell}} C_{\ell jt}^{\frac{\sigma_\ell-1}{\alpha_\ell}}
\]

and \( \delta(u_t) = (1/\tau)u_t^\tau, \tau > 1 \).

After a few substitution the problem reduces to the following

\[
\max_{\{C_{it}, C_{ijt}, N_{it}, M_{jt}, K_{jt}, K_{t+1}, u_t \}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t \left\{ U_h(C_{ht}, N_{ht}) + U_\ell(C_{\ell t}, N_{\ell t}) \right. \\
+ \mu_1, t \left[ A_{gt} K_{gt}^{\frac{1}{\alpha_g}} N_{ht}^{\alpha_g} - M_{gt} - C_{hgt} - \eta_h C_{hht} - C_{\ell gt} - \eta_\ell C_{\ell ht} \right] \\
+ \mu_2, t \left[ A_{st} K_{st}^{\frac{1}{\alpha_s}} N_{\ell t}^{\alpha_s} - M_{st} - C_{hmt} - C_{\ell mt} \right] \\
+ \mu_3, t \left[ \left( \omega_{g1} M_{gt}^{\frac{\sigma_g-1}{\alpha_g}} + \omega_{s1} M_{st}^{\frac{\sigma_s-1}{\alpha_s}} \right)^{\frac{1}{\sigma_g}} \right] + K_{t+1} + \left[ 1 - \delta(u_t) \right] \frac{K_t}{q_t} \right. \\
+ \mu_4, t \left[ u_t K_t - K_{gt} - K_{st} \right] \\
+ \mu_5, t \left[ \sum_{j \in \{h,m\}} \Omega_{hj}^{\frac{1}{\alpha_h}} C_{hst}^{\frac{1}{\alpha_h}} C_{hjt}^{\frac{\sigma_h-1}{\alpha_h}} \right] \\
+ \mu_6, t \left[ \sum_{j \in \{h,m\}} \Omega_{\ell j}^{\frac{1}{\alpha_\ell}} C_{\ell st}^{\frac{1}{\alpha_\ell}} C_{\ell jt}^{\frac{\sigma_\ell-1}{\alpha_\ell}} \right]
\]

And using the optimality conditions for Leontief preferences

\[
C_{hst} = C_{ht}, \quad C_{hgt} = \xi_h C_{hht}^{\nu_h}
\]

\[
C_{\ell st} = C_{\ell t}, \quad C_{\ell gt} = \xi_\ell C_{\ell ht}^{\nu_\ell}
\]
we obtain the following simpler problem:

\[
\max_{\{C_{it}, C_{ijt}, N_{it}, M_{jt}, K_{jt}, K_{t+1}, u_t\} \in \{\ell, h\}, j \in \{g, h, m\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ U_h(C_{ht}, N_{ht}) + U_{\ell}(C_{\ell t}, N_{\ell t}) \right. \\
+ \mu_{1,t} \left[ A_{gt} K_{gt}^{1-\alpha_g} N_{ht}^{\alpha_g} - M_{gt} - \xi_\ell^{1/\nu_h} C_{ht}^{1/\nu_h} - \eta_{ht} C_{ht} - \xi_\ell^{1/\nu_{ht}} C_{\ell t}^{1/\nu_{ht}} - \eta_{\ell ht} C_{\ell ht} \right] \\
+ \mu_{2,t} \left[ A_{st} K_{st}^{1-\alpha_s} N_{\ell t}^{\alpha_s} - M_{st} - C_{ht} C_{\ell t} \right] \\
+ \mu_{3,t} \left[ \left( \frac{1}{\omega_g^{1/\nu_{gt}}} M_{gt}^{1-\nu_h} + \frac{1}{\omega_s^{1/\nu_{st}}} M_{st}^{1-\nu_{ht}} \right)^{\nu_{ht} - 1} - \frac{K_{t+1}}{q_t} + \frac{1}{q_t} \right] \\
+ \mu_{4,t} \left[ u_t K_t - K_{gt} - K_{st} \right] \\
+ \mu_{5,t} \left[ \sum_{j \in \{h, m\}} \Omega_{hj}^{1/\nu_h} C_{ht}^{\nu_h-\sigma_h} C_{ht}^{\sigma_h-1} \right] \\
+ \mu_{6,t} \left[ \sum_{j \in \{h, m\}} \Omega_{\ell j}^{1/\nu_{\ell t}} C_{\ell t}^{\nu_{\ell t}-\sigma_{\ell t}} C_{\ell t}^{\sigma_{\ell t}-1} \right] \left\} \right.
\]

39
The first order conditions (for an interior optimum) are:

\[
C_{ht} : -\Phi_{1h}U_{ht}^{\gamma} + \frac{1}{\nu_h} \mu_{1h} C_{ht}^{\nu_h - 1} = \mu_{5t} \left[ \left( \frac{\epsilon_{hh} - \sigma}{\sigma} \right) \Omega_{hh}^2 \frac{C_{ht}^{\nu_h - 1}}{C_{ht}^{\sigma}} + \left( \frac{\epsilon_{hm} - \sigma}{\sigma} \right) \Omega_{hm}^2 \frac{C_{ht}^{\nu_h - 1}}{C_{ht}^{\sigma}} \right]
\]

\[
C_{hht} : \eta_h \mu_{1t} = \mu_{5t} \left( \frac{\sigma - 1}{\sigma} \right) \Omega_{hh}^1 \frac{C_{ht}^{\nu_h - 1}}{C_{ht}^{\sigma}}
\]

\[
C_{hmt} : \mu_{2t} = \mu_{5t} \left( \frac{\sigma - 1}{\sigma} \right) \Omega_{hm}^1 \frac{C_{ht}^{\nu_h - 1}}{C_{ht}^{\sigma}}
\]

\[
C_{lt} : -\Phi_{1l}U_{lt}^{\gamma} + \frac{1}{\nu_t} \mu_{1l} C_{lt}^{\nu_t - 1} = \mu_{6t} \left[ \left( \frac{\epsilon_{lt} - \sigma}{\sigma} \right) \Omega_{lt}^2 \frac{C_{lt}^{\nu_t - 1}}{C_{lt}^{\sigma}} + \left( \frac{\epsilon_{ltm} - \sigma}{\sigma} \right) \Omega_{ltm}^2 \frac{C_{lt}^{\nu_t - 1}}{C_{lt}^{\sigma}} \right]
\]

\[
C_{lth} : \eta_l \mu_{1t} = \mu_{6t} \left( \frac{\sigma - 1}{\sigma} \right) \Omega_{lt}^1 \frac{C_{lt}^{\nu_t - 1}}{C_{lt}^{\sigma}}
\]

\[
C_{ltm} : \mu_{2t} = \mu_{6t} \left( \frac{\sigma - 1}{\sigma} \right) \Omega_{ltm}^1 \frac{C_{lt}^{\nu_t - 1}}{C_{lt}^{\sigma}}
\]

\[
N_{ht} : \Phi_{1h} \Phi_{2h} U_{ht}^{\gamma} N_{ht}^{\theta_h} = \mu_{1t} \alpha_g Y_{gt} N_{ht}^{\theta_t - 1}
\]

\[
N_{lt} : \Phi_{1l} \Phi_{2l} U_{lt}^{\gamma} N_{lt}^{\theta_t} = \mu_{2t} \alpha_s Y_{st} N_{lt}^{\theta_t - 1}
\]

\[
M_{gt} : \mu_{1t} = \mu_{3t} A_{lt} \left( \frac{1}{\omega_g} M_{gt}^{c^{-1}} + \frac{1}{\omega_s} M_{st}^{c^{-1}} \right) \frac{\epsilon^{c-1}}{\epsilon^1} \frac{1}{\omega_g} M_{gt}^{c^{-1}}
\]

\[
M_{st} : \mu_{2t} = \mu_{3t} A_{lt} \left( \frac{1}{\omega_g} M_{gt}^{c^{-1}} + \frac{1}{\omega_s} M_{st}^{c^{-1}} \right) \frac{\epsilon^{c-1}}{\epsilon^1} \frac{1}{\omega_s} M_{st}^{c^{-1}}
\]

\[
K_{gt} : \mu_{1t} (1 - \alpha_g) Y_{gt} K_{gt}^{-1} = \mu_{4t}
\]

\[
K_{st} : \mu_{2t} (1 - \alpha_s) Y_{st} K_{st}^{-1} = \mu_{4t}
\]

\[
u_t : \mu_{3t} \frac{1}{q_t} = \mu_{4t}
\]

\[
K_{t+1} : \mu_{3t+1} \frac{1}{q_{t+1}} = \beta \mathbb{E}_t \left[ \mu_{3,t+1} \left( \frac{1 - \delta(u_{t+1})}{q_{t+1}} \right) + \mu_{4,t+1} u_{t+1} \right]
\]

Combining the last two equations we obtain

\[
\mu_{3t} \frac{1}{q_t} = \beta \mathbb{E}_t \left[ \mu_{3,t+1} \left( \frac{1 - \delta(u_{t+1})}{q_{t+1}} \right) + \mu_{4,t+1} u_{t+1} \right]
\]

\[
\frac{\mu_{4t}}{\delta'(u_t)} = \beta \mathbb{E}_t \left[ \frac{\mu_{4,t+1} q_{t+1}}{\delta'(u_{t+1})} \left( \frac{1 - \delta(u_{t+1})}{q_{t+1}} \right) + \mu_{4,t+1} u_{t+1} \right]
\]

\[
\frac{\mu_{4t}}{u_{t+1}} = \beta \mathbb{E}_t \frac{\mu_{4,t+1}}{u_{t+1}} \left[ 1 + \left( \frac{1}{\tau} \right) u_{t+1} \right]
\]

\[
\frac{\mu_{1t}}{u_{t+1}} = \beta \mathbb{E}_t \mu_{1,t+1} \frac{Y_{gt} K_{gt}^{-1}}{u_{t+1}^{\gamma - 1}} \left[ 1 + \left( \frac{1}{\tau} \right) u_{t+1} \right]
\]

40
Rearranging the rest of the FOCs, we obtain the following

\[ \Phi_1 \Phi_2 \Gamma^{-1} \Gamma_2 \Gamma_1 \approx \mu_1 t \alpha_g Y_{gt} \]
\[ \Phi_1 \Phi_2 \Gamma^{-1} \Gamma_2 \Gamma_1 \approx \mu_2 t \alpha_s Y_{st} \]

\[ \mu_1 t = \left[ \frac{\omega_g M_{st}}{\omega_s M_{gt}} \right]^{\frac{1}{2}} \]
\[ \mu_1 t \mu_2 t M_{gt} + \mu_2 t M_{st} = \mu_3 t I_t \]

\[ \mu_1 t (1 - \alpha_g) Y_{gt} + \mu_2 t (1 - \alpha_s) Y_{st} = \mu_4 t u_t K_t \]

\[ \mu_3 t - \mu_4 t = \mu_4 t \]

\[ \mu_1 t - \mu_2 t Y_{gt} K_{gt}^{-1} \frac{1}{u_t^{-1}} = \beta \varepsilon_t \mu_1, t+1 \frac{Y_{gt+1} K_{gt+1}^{-1}}{u_{t+1}^{-1}} \left[ 1 + \left( \frac{1}{\tau} \right) u_{t+1}^{-1} \right] \]

Finally, work on the second last equation in the system:

\[ \frac{\mu_3 t}{\mu_2 t} \delta'(u_t) \frac{1}{q_t} \left( \frac{\mu_1 t}{\mu_2 t} M_{gt} + M_{st} \right) = \frac{1}{u_t K_t} \left( \frac{\mu_1 t}{\mu_2 t} (1 - \alpha_g) Y_{gt} + (1 - \alpha_s) Y_{st} \right) \]

\[ \frac{\delta'(u_t)}{q_t} \left( \frac{1}{I_t} \left( \frac{\beta q_t}{\omega_s M_{gt}} \right)^{\frac{1}{2}} M_{gt} + M_{st} \right) = \frac{1}{u_t K_t} \left( \frac{(1 - \alpha_s) Y_{st} K_{gt} (1 - \alpha_g) Y_{gt} + (1 - \alpha_s) Y_{st}}{(1 - \alpha_s) Y_{st}} \right) \]

\[ \frac{\delta'(u_t)}{q_t} \left( \frac{1}{I_t} \frac{[M_{st}]}{\omega_s I_t} \right)^{\frac{1}{2}} \left( \frac{I_t}{A_{it}} \right)^{\frac{1}{2} - 1} = \frac{1}{u_t K_t} \left( \frac{(1 - \alpha_s) Y_{st} K_{gt} + (1 - \alpha_s) Y_{st}}{K_{st}} \right) \]

\[ \frac{u_t^{-1}}{q_t} \left[ \frac{M_{st}}{\omega_s I_t} \right]^{\frac{1}{2}} A_{it}^{1 - e} = \frac{(1 - \alpha_s) Y_{st}}{K_{st}} \]
The complete system (together with definitions) reads:

\[-\Phi_1 h \mathcal{U}_{ht}^{-\gamma} + \frac{\xi_t}{\nu_t} \mu_1 t C_{ht}^{1-\gamma} = \eta_h \left( \frac{\epsilon_{ht} - \sigma_h}{\sigma_h - 1} \right) \left( \frac{C_{ht}}{C_{ht}} \right) \mu_1 t + \left( \frac{\epsilon_{hm} - \sigma_h}{\sigma_h - 1} \right) \left( \frac{C_{hmt}}{C_{ht}} \right) \mu_2 t \]

\[-\Phi_1 e \mathcal{U}_{et}^{-\gamma} + \frac{\xi_e}{\nu_e} \mu_1 t C_{et}^{1-\gamma} = \eta_e \left( \frac{\epsilon_{et} - \sigma_e}{\sigma_e - 1} \right) \left( \frac{C_{et}}{C_{et}} \right) \mu_1 t + \left( \frac{\epsilon_{elt} - \sigma_e}{\sigma_e - 1} \right) \left( \frac{C_{elt}}{C_{et}} \right) \mu_2 t \]

\[\Phi_1 h \Phi_2 h \mathcal{U}_{ht}^{-\gamma} N_{ht}^{1+\theta_h} = \mu_{1t} \alpha_g Y_{gt} \]

\[\Phi_1 e \Phi_2 e \mathcal{U}_{et}^{-\gamma} N_{et}^{1+\theta_e} = \mu_{2t} \alpha_s Y_{st} \]

\[\frac{u_t^{\gamma-1}}{q_t} \left[ \frac{M_{st}}{\omega_s I_t} \right]^{\frac{1}{\alpha_s}} A_{t+1}^{\frac{1-\alpha_s}{\alpha_s}} = \frac{(1 - \alpha_s) Y_{st}}{K_{st}} \]

\[\mu_{1t} \left( \frac{Y_{gt} K_{gt}^{-1}}{u_t^{\gamma-1}} \right) = \beta E \mu_{1,t+1} \left( \frac{Y_{g,t+1} K_{g,t+1}^{-1}}{u_t^{\gamma-1}} \right) \left[ 1 + (1 - \frac{1}{\tau}) u_t^{\gamma-1} \right] \]

\[Y_{gt} = M_{gt} + \xi_h \frac{1}{\nu_h} C_{ht}^{1-\gamma} + \eta_h C_{hht} + \xi_e \frac{1}{\nu_e} C_{et}^{1-\gamma} + \eta_e C_{tht} \]

\[Y_{st} = M_{st} + C_{hmt} + C_{elt} \]

\[I_t = \frac{K_{t+1}}{q_t} - [1 - \delta(u_t)] \frac{K_t}{q_t} \]

\[u_t K_t = K_{gt} + K_{st} \]

\[Y_{gt} = A_{gt} K_{gt}^{1-\alpha_g} N_{ht}^{\alpha_g} \]

\[Y_{st} = A_{st} K_{st}^{1-\alpha_s} N_{et}^{\alpha_s} \]

\[I_t = A_{t+1} \left( \frac{1}{\omega_s} M_{gt}^{\gamma-1} + \frac{1}{\omega_h} M_{st}^{\gamma-1} \right) \]

\[1 = \sum_{i \in \{g, s\}} \Omega_{hi}^{\sigma_h} C_{hi}^{\sigma_h} C_{hi}^{\sigma_h^{-1}} \]

\[1 = \sum_{i \in \{g, s\}} \Omega_{ti}^{\sigma_t} C_{ti}^{\sigma_t} C_{ti}^{\sigma_t^{-1}} \]

There are 19 equations and 19 variables:

\[C_h, C_{hh}, C_{hm}, C_{et}, C_{et}, C_{elt}, N_h, N_e, u, M_g, M_s, K_g, K_s, K, Y_g, Y_s, I, \mu_1, \mu_2 \]
Computing GDP

The social planner’s solution can be decentralized by the following pair of prices

\[ P_{gt} = A_t^{-\frac{1}{\epsilon}} \left( \frac{\omega_g I_t}{M_{gt}} \right)^{\frac{1}{\epsilon}} \]
\[ P_{st} = A_t^{-\frac{1}{\epsilon}} \left( \frac{\omega_s I_t}{M_{st}} \right)^{\frac{1}{\epsilon}} \]
\[ R_t = (1 - \alpha_g) P_{gt} Y_{gt} / K_{gt} \]
\[ w_{ht} = \alpha_g P_{gt} Y_{gt} / N_{ht} \]
\[ w_{lt} = \alpha_s P_{st} Y_{st} / N_{lt} . \]

In order to recover prices it is enough to focus on the problem of the firm producing investment good. First recall that we had obtained the following expressions

\[ M_{gt} = A_t^{-1} \omega_g P_{gt}^{-1} I_t \]
\[ M_{st} = A_t^{-1} \omega_s P_{st}^{-1} I_t . \]

From these two we can solve for \( P_{gt} \) and \( P_{st} \).

Similarly, from the problem of the firms producing basic goods and service goods, we obtain the wages and the rental rate of capital.

Finally, after obtaining the prices in a competitive equilibrium we can define GDP as the total value of consumption and investment expressed in terms of the investment good:

\[ Y_t = P_{gt} C_{gt} + P_{st} C_{st} + I_t \]

where

\[ C_{gt} = C_{hgt} + C_{ltg} \]
\[ C_{st} = C_{hst} + C_{lst} \]
Appendix 4

The planner’s problem with incomplete specialization is to find a pair of welfare weights \( \{ \Phi_{1\ell}, \Phi_{1h} \} \) and a sequence of contingency plans \( \{ C_{jgt}, C_{jhht}, C_{jmt}, N_{jt} \} \) for type \( j \in \{ h, \ell \} \) consumers and for aggregate allocations \( \{ K_t, K_{gt}, K_{st}, I_t, M_{gt}, M_{st}, u_t, s_t \} \) such that the sum of individual utilities that are weighted by the corresponding welfare weights is maximized:

\[
\max_{\{C_{it}, C_{jgt}, N_{it}, M_{jt}, K_{jt}, K_{jt+1}, u_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ m_t U_h(C_{ht}, N_{ht}) + (1 - m_t) U_l(C_{lt}, N_{lt}) + \mu_{1,t} \left[ Y_{gt} - M_{gt} - m_t(C_{ht} + \eta_h C_{hht}) - (1 - m_t)(C_{lt} - \eta_l C_{lht}) \right] + \mu_{2,t} \left[ Y_{st} - M_{st} - m_t(C_{hmt} - (1 - m_t)C_{lmt}) \right] + \mu_{3,t} \left[ \left( \frac{1}{\omega_g M_{gt}^{\omega}} + \frac{1}{\omega_s M_{st}^{\omega}} \right) - \frac{K_{jt+1}}{q_t} + [1 - \delta(u_t)] \frac{K_t}{q_t} \right] + \mu_{4,t} [u_t K_t - K_{gt} - K_{st}] + \mu_{5,t} \left[ \sum_{j \in \{h,m\}} \Omega_{hj}^{\frac{\sigma_h - \sigma_j}{\sigma_h}} C_{hjt}^{\frac{\sigma_h - 1}{\sigma_h}} - 1 \right] + \mu_{6,t} \left[ \sum_{j \in \{h,m\}} \Omega_{lj}^{\frac{\sigma_j - \sigma_l}{\sigma_j}} C_{ljt}^{\frac{\sigma_j - 1}{\sigma_j}} - 1 \right] \right\}
\]

where

\[
Y_{gt} = A_{gt} K_{gt}^{1 - \alpha_g} N_{ht}^{\alpha_g} \\
Y_{st} = A_{st} K_{st}^{1 - \alpha_s} N_{lt}^{\alpha_s} \\
N_{ht} = \chi_h \int_{s_t}^{\infty} e_h(s) f(s) \, ds = \chi_h (1 + \lambda s_t) e^{-\lambda s_t} \\
N_{lt} = \chi_l \int_{0}^{s_t} f(s) \, ds = \chi_l (1 - e^{-\lambda s_t}) \\
m_t = e^{-\lambda s_t} \\
\delta(u_t) = (1/\tau) u_t^\tau, \quad \tau > 1.
\]

and \( m_t = 1 - F(s_t) \) is the measure of high skill agents and \( \delta(u_t) = (1/\tau) u_t^\tau \) is an increasing, convex function with \( \tau > 1 \).