Non-homothetic Preferences, Division of Labor and the Business Cycle*

Job Market Paper

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Abstract

This paper introduces a new internal propagation mechanism into a baseline multi-sector RBC model by focusing on the cross-sectional interdependencies across the population that are generated by the discrepancy between individuals' consumption behavior and their production activities. In the model, there are two types of agents (L and H-skill) who differ in terms of the occupations that they are specialized in and the composition of their consumption baskets. In particular, both types of agents have non-homothetic preferences which induce a strong desire to consume the good that the other type produces. The resulting mechanism, in response to a positive investment shock, works as follows: as H-skill agents are employed in a more capital-intensive sector, their productivities, and so their incomes, initially increase relatively more compared to the L-skill agents. But they spend their additional income mostly on service goods in which L-skill agents are specialized in, so the demand for L-skill labor subsequently increases as well. On the other hand, as their income rises, L-skill agents start consuming relatively more capital intensive goods, which in turn generates additional demand for H-skill labor that is not related to the original technology shock. Consequently, this mechanism feeds into itself and a circular interaction between the two types of agents emerges as a result of the imposed preference structure and the division of labor. I show, through a series of quantitative experiments, that this interaction serves as a very strong internal propagation mechanism that can significantly increase the amplification of exogenous shocks- a long quest in the RBC literature. I also discuss the way the same mechanism gives rise to an endogenous variation in the capital content of the goods consumed over the business cycle and how this endogenous variation generates more persistence in the model.

JEL Classification: E32, E30, E25

Keywords: Business Cycles, Multi-sector Model, Non-homothetic Preferences, Division of Labor.

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1 Introduction

In this paper, I consider an economy in which agents are specialized either in goods or service producing sectors and have different preferences over the goods produced in these two sectors. In this framework, specialization leads naturally to the set of goods which agents contribute to its production and their consumption bundles being different. This discrepancy between the consumption bundles and production activities in return introduces a new type of interdependency among the agents across the population, in addition to one that already present in general equilibrium models with homogeneous agents. Here, it arises as a result of *the functional roles assumed through the division of labor* in the economy. In this environment, I study how the demand for the type of labor an agent supplies and her consumption behavior interacts with each other in propagating the exogenous economic shocks.

In order to motivate the framework the model is set in and some of its assumptions, I will discuss, as an example, how work and leisure decisions of different agents are in fact related across the population. The type of interdependency that I mention above arises as soon as *leisure activities*[1] are modeled as commodities rather than considering leisure as a fraction of the time endowment that agents can *enjoy by doing nothing* as modeled in the standard models. For example, according to the time-use surveys, the time allocated to watching T.V. is one of the main leisure activities (Aguiar and Hurst (2008)). But it is hard to argue that those who watch more T.V. derive more utility compared to those who do not watch much T.V. but instead utilize their time in different activities. Indeed, the same paper documents that in comparison to less educated men, more educated men spend less time on T.V., but more on other leisure activities like exercising, reading, and hobbies. Moreover, Aguiar and Hurst (2016), after showing that lower skilled individuals increased their leisure time between 1985 and 2003 while higher skilled individuals lowered their leisure time, they conclude: *The increase in leisure inequality has matched the well-documented increase in income and consumption inequality during the last 30 years documented by many in the literature.*

[1] Leisure activities include time spent watching television, socializing, going to the movies, playing video games, exercising, and sleeping (see Aguiar and Hurst (2016)).

Figure 1 depicts the total weekly leisure choices by educational attainment and gender over the last three decades. The figure shows that both for men and women the total leisure time taken decreases as the level of education rises.[2] It is hard to rationalize these patterns observed in the data if leisure is assumed to be a normal good just as any other consumption good. Otherwise, higher skilled individuals, who have had less time for *doing nothing* over the last 30 years, could simply imitate the lifestyle of lower skilled individuals. However, they do not and their lifestyle would be considered as more desirable.[3] Therefore, I deemphasize the choice between work and leisure. Instead, I consider chosen consumption bundles and work decisions as pairs that are *jointly revealed* in the data as an outcome of some rational preferences over the set of feasible pairs which, I assume, are determined outside the model. In the previous T.V. example, I consider watching more T.V. as a revealed preference outcome given the feasible alternatives and usually it is more common among the lower skilled people for whom better options are *not feasible*.

In order to see how work-leisure decisions are interdependent in the model, let us look at what might happen after a positive technology shock: higher skilled individuals become more productive and they are inclined to buy more time-saving goods (like eating out or hiring a dog walker, etc.) to be able to work more in the saved time and buy higher intensity leisure activities (going to a Broadway show rather than going to the cinema or watching T.V. at home). Since other agents are specialized in the production of these time-saving jobs or higher intensity leisure activities, demand for their labor increases as well, even though the technology shock had no immediate impact on their productivity in the jobs that they are specialized in. In the model, this mechanism actively contributes to the propagation of the economic shocks. [4]

Going back to the work-leisure choice discussion, in many cases, once an occupation/role is taken, there is not much flexibility to adjust the time spent at work. Think of someone who is about to decide between an academic job versus working in finance. These two professions offer two very different lifestyles, but the agent knows this at the time of decision and once

[2] The figure also shows that the gap between the total leisure time taken by high and low skilled individuals has significantly increased over the last thirty years.

[3] For lower skilled agents imitating the lifestyle of higher skilled individuals simply might not be feasible.

[4] Also note that, this mechanism might be particularly important in capturing the movements in the labor market at the extensive margin over the business cycle, which has been considered to be more important (see Hansen (1985)) compared to adjustments at the intensive margin.

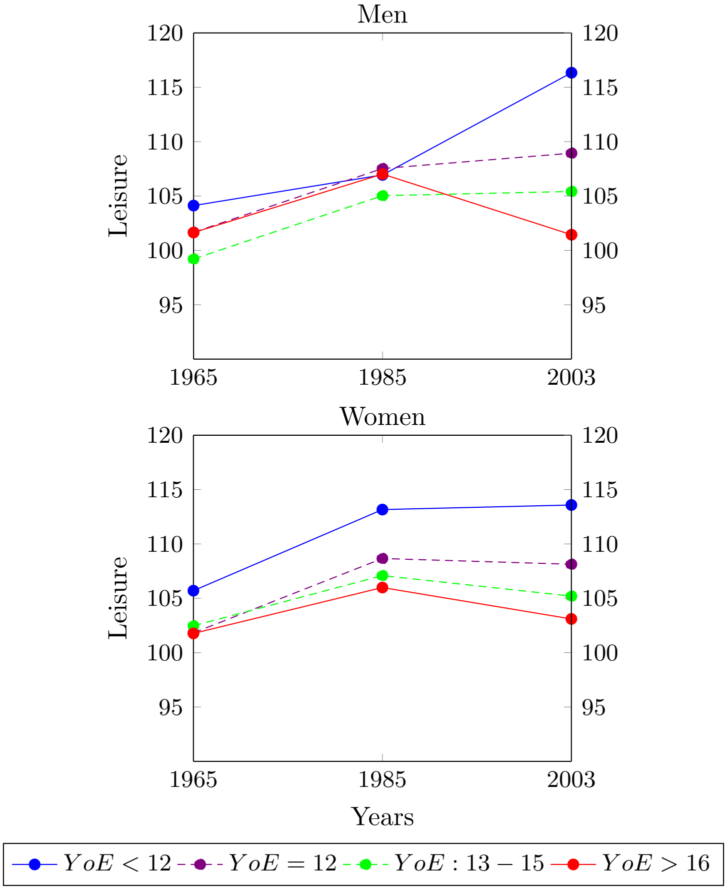

Figure 1: Leisure by Years of Education (YoE). Source: Comprised from Aguiar and Hurst (2007), Table V.

she chooses her profession she just fulfills her typical expected role in that lifestyle. Of course, in the same job, different people might end up with different amounts of time spent at work, but since we are interested in averages across consumers with similar characteristics these heterogeneities wash out within the group.

In summary, in this framework, the decision between working time and leisure is not as central as in the standard models. Instead, agents are modeled as if they chose a lifestyle given their skill endowments and then in response to the exogenous shocks, they adjust their lifestyle (consumption behavior in the model) as much as possible under the relevant economic constraints.

The model features a novel amplification mechanism that arises as a result of the combined effects of the differential factor contents of the goods consumed by different agents as well as the preference structure imposed. In particular, I assume that agents prioritize increasing the fraction of basic goods in their consumption bundles up to a certain level, and then the propensity to spend on market produced services increases, so the preferences are non-homothetic. The opposite is true for higher skilled individuals; their consumption bundles already contain a satisfactory level of goods so they expand their consumption towards services faster. Now since goods and services have different capital intensities, this preference structure gives rise to an endogenous variation in the average capital intensity of the goods consumed over the business cycle. Since the capital stock is the main driver and the only stock variable in the model, an endogenous variation in the average capital intensity of the goods consumed has first order effects on the statistical properties of the model. I show that under the maintained assumptions the model exhibits significant gains in terms of persistence and amplification of the shocks.

In the next sections, after a brief literature survey, first I present the model with complete specialization and then assess its performance through a series of quantitative experiments, and finally in the last section I extend the model to allow incomplete specialization.

2 Related Literature

In this section, I discuss how my model is related to several strands of the business cycle literature.

Home production models (Benhabib, Rogerson and Wright (1991)) are based on a related observation that there is a high degree of cyclicality in market consumption of goods and services that are substitutes for home production (such as eating at restaurants, house-cleaning, child-care, etc). These models try to account for this observation by modeling the trade-off that agents face when they allocate their total time endowment between market and non-market activities over the business cycle, which in return, are driven primarily by changes in the market technology relative to the home technology. But since there is no specialization in these models, the circular interaction that I am after in this chapter does not appear in these models.

Chang (2000) and Bils, Chang and Kim (2012) proposes business cycle models with occupational choice based on comparative advantage. In contrast to these papers, in my model there are two types of agents with different consumption behaviors and the sectors differ in terms of their factor intensities.

Models of industrial input-output networks (Dupor (1999), Horvath (2000), Acemoglu et al. (2012)) focus on the production side of the economy and typically impose a network structure among the sectors and allow sectoral heterogeneity in terms of their capital intensities. In contrast to these papers, in my model the demand side of the economy receives more attention in the modeling and its interaction with the production structure in propagating the exogenous shocks is the central question I would like to investigate.

Similar to the business cycle models with investment specific shocks (Greenwood, Hercowitz and Huffman (1988), Greenwood, Hercowitz and Krusell (1997), Greenwood, Hercowitz and Krusell (2000)), I assume that exogenous shocks effect the marginal efficiency of investment but not existing capital and this generates a wedge in the user cost of capital across time, and firms respond to this wedge by adjusting their capacity utilizations.

In a recent paper, Jaimovich, Rebelo and Wong (2017) allow households to choose both the quantity and quality of the goods they consume and show that the choice of the quality of the goods consumed affects the total employment significantly. They assume that the labor intensity increases with the quality, i.e. the production of higher quality goods contain more labor. On the other hand, in my model, I emphasize the functional roles assumed in an economy based on the division of labor, which in return requires to allow a more flexible relationship between labor intensity and quality. For example, in my model, in contrast to their assumption, market produced service goods are more labor intensive but they do not necessarily offer higher utility for all individuals.

Galı (1994) and Benhabib and Wen (2004) proposes models with demand shocks and emphasize the compositional effects of demand on the dynamics of output as in my model. In more related work, Benhabib, Perli and Sakellaris (2006) discusses in detail how compositional effects may play a significant role in the dynamic properties of aggregate variables in the two, especially in the three-sector real business cycle models provided that the factor intensities among the sectors are different enough. However, in their model there is a single consumption good and their main focus is to produce the right output dynamics, which they define as hump-shaped impulse response functions and positive autocorrelation coefficients for at least a few periods after the shock.

Finally, this paper addresses some of the issues, most notably raised by Cogley and Nason (1995), related to performance of the real business cycle models. Cogley and Nason (1995) conclude that *many RBC models have weak internal propagation mechanisms and do not generate interesting dynamics via their internal structure.* As they mention in their paper, these models rely on two main propagation mechanisms:[5] capital accumulation and intertemporal substitution. On the one hand, as the intertemporal substitution of labor-leisure and capital accumulation are flip coins of the same story in these models they can deliver an elegant story parsimoniously, but on the other hand, the same exact feature puts the modeler into a conceptual straitjacket. In this respect, the present model is just another attempt to build a business cycle model with a stronger internal propagation mechanism that arises as a result of the additional features mentioned above.

3 Preferences

3.1 A Discussion of Consumption Categories

As discussed in the introduction, instead of making the labor-leisure choice a central point in the modeling, I model agents as if they choose an optimal consumption bundle over

[5] I skip the mechanisms that are based on adjustment costs or lags.

different goods as a function of their relative positions in the economy (their wage income in my model). Of course, the set of possible consumption goods that individuals choose from can be specified in many different ways but, I think in terms of its relevance to the business cycle research, the following two group categorization over goods and services[6] is particularly useful in investigating interdependencies generated by the division of labor across the population.

(i) Category of Goods (C_g): This category includes all the goods that *do not require* someone else's labor service at the moment of consumption. However, they might be of various qualities. For simplicity of the exposition in this section, assume they come only in two versions; goods are either basic goods (C_b) or customized goods (C_c). The main difference between these two versions of goods is their skill intensities, the latter has a greater high-skill content.

(ii) Category of Services (C_s): On the other hand, goods consumed under C_s should satisfy the following condition: they can potentially be met by the consumer themselves once the necessary inputs are purchased at the market, and if they are not met by the consumer, they *require* someone else's labor service at the time of consumption. In the former case, the consumer produces the service at home but has to decide between the basic or customized inputs, C_b and C_c, to produce the service goods at home (C_h). On the other hand, in the latter case the consumer purchases the service from the market (C_m), which is produced by firms using again C_b and C_c and labor, in addition. For instance, consider someone who would like to learn to play the guitar. If a computer software is purchased for that, the service is C_h, and if a music teacher is hired the service consumed is C_m.

In summary, consumers derive utility from goods (C_g) and services (C_s) according to a consumption aggregate

$$C_t = C(C_{gt}, C_{st}),$$

and in return both C_{gt} and C_{st} have two versions:

$$C_{gt} \in \{C_{bt}, C_{ct}\}, \quad C_{st} \in \{C_{ht}, C_{mt}\},$$

[6]The distinction between goods and services is standard, but the way that I define these two categories is more specific.

that is, agents can consume either the basic version of the goods (C_{bt}) or the customized version (C_{ct}). Similarly, agents choose between market produced service goods, C_{mt}, and home produced service goods, C_{ht}.

In the following sections, I assume that only the basic good (C_{bt}) is produced from the goods category and the services category contain two versions as mentioned above.

3.2 Specification of Utility Functions

I assume that the preferences over goods and service categories are represented by the following Leontief function

$$C_t = \min\left\{C_{st},\, \xi C_{gt}^{\nu}\right\}$$

Although this utility function implies that the elasticity of substitution between goods and services is zero, because of the nonlinearity in the second term, consumers expand their consumption bundles in favor of service goods as the consumption index (or real income) rises. The parameter ν controls how fast consumers switch from goods to services as their real income rises.[7] Therefore this particular specification of Leontief function leads to nonlinear income expansion (Engel) curves. More specifically, when $\nu > 1$ this utility function implies that both goods are normal goods and service goods is a luxury good. In Figure 2 I plot the indifference curves and the income expansion path corresponding to $\xi = 1$ and $\nu = 2$. This view is consistent with the empirical results discussed in subsequent sections. Although this assumption might be extreme and partly made for the sake of simplicity, I think it is also a reasonable assumption for the following reasons. First, my main concern in this paper is to evaluate the contribution of endogenous variation in the average factor content of demand to the business cycle properties, not how endogenous variation in relative prices effect the allocation of resources between the consumption categories. Second, related to the first point, I do not have any theory for the variation of relative prices but empirically there is an ample evidence supporting the pattern depicted in Figure 2, so I consider this specification as an empirically supported *discipline* that will impose restrictions

[7]Mathematically, $(\partial C_{st}/\partial C_{gt})\,(C_{gt}/C_{st}) = \nu$ therefore as the consumer moves across the indifference curves the composition of his consumption basket changes at a constant rate ν.

Figure 2: A Nonlinear Income Expansion Path and Indifference Curves ($\nu = 2$)

over how the other endogenous variables should behave in the model when the effect of relative prices is eliminated.[8]

For the two goods, C_{ht} and C_{mt}, consumed under service category, I assume that the preferences are *implicitly* defined by the following isoelastic specification

$$1 = \sum_{j\in\{h,m\}}^{I} \Omega_j^{\frac{1}{\sigma}} C_{st}^{\frac{\epsilon_j-\sigma}{\sigma}} C_{jt}^{\frac{\sigma-1}{\sigma}} . \tag{3.2.1}$$

These preferences are non-homothetic generalizations of CES preferences.[9] They were originally proposed in mid-seventies by Sato (1975) and Hanoch (1975) and they have recently been used in a few papers in the structural change literature (Comin, Lashkari and Mestieri (2015), Matsuyama (2017), Duernecker, Herrendorf and Valentinyi (2016), see also Appendix 1). The parameters in the utility function have the following interpretations: σ measures the elasticity of substitution between home produced services (C_{ht}) and market provided services C_{mt}; ϵ_j measures the income sensitivity of the share of expenditures on

[8]To clarify, Leontief functional form eliminates the relative price effect.

[9]When $\epsilon_h = \epsilon_m = 1$, these preferences reduce to the standard CES preferences

$$C_{st} = \left(\sum_{j\in\{h,m\}}^{I} \Omega_j^{\frac{1}{\sigma}} C_{jt}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

the good C_{jt}, $j \in \{h, m\}$. Through this non-homothetic CES specification, one can obtain independent income and substitution elasticities for market and home produced goods. In particular, Comin, Lashkari and Mestieri (2015) show that the demand functions resulting from these preferences have the following two distinctive properties:

1. The elasticity of the relative demand for home and market produced services with respect to the aggregate consumption index for the services is constant, i.e.,

$$\frac{\partial \log(C_{ht}/C_{mt})}{\partial \log C_{st}} = \epsilon_h - \epsilon_m.$$

2. The elasticity of substitution between home and market produced services is uniquely defined and constant

$$\frac{\partial \log(C_{ht}/C_{mt})}{\partial \log(P_{ht}/P_{mt})} = \sigma.$$

As they discuss, the first property ensures that the non-homothetic features of these preferences do not systematically vary as income grows, in contrast to, for example, Stone-Geary preferences. On the other hand, the second property ensures that the patterns of intra-sectoral substitution have a constant price elasticity and, thus, do not systematically vary as income grows. I find the latter property particularly useful since, as stated above, I have no theory for the evolution of relative prices over the business cycle, these preferences ensures that at least the demand side of the economy alone will not lead to the variations in the relative prices. I interpret this parameter σ as the consumer's willingness to diversify its consumption expenditures. As I will discuss in the next section, household spending becomes more diversified as income rises, which in return suggest that high income households should have a higher elasticity of substitution compared to lower income households ($\sigma_h > \sigma_\ell$) as will be the case in the quantitative exercises.

Finally, to represent the preferences over consumption and leisure I use preferences with no intertemporal wealth effect on the labor supply as in Greenwood, Hercowitz and Huffman (1988) (GHH henceforth).

$$U(C_{gt}, C_{ht}, C_{mt}N_{ht}) = \frac{\Phi_1}{1-\gamma}\left(C_t - \Phi_2 \frac{N_t^{1+\theta}}{1+\theta}\right)^{1-\gamma} \tag{3.2.2}$$

where C_t is implicitly defined by the following relationships

$$C_t = \min\left\{C_{st},\, \xi C_{gt}^{\nu}\right\} \tag{3.2.3}$$

$$1 = \sum_{j\in\{h,m\}}^{I} \Omega_j^{\frac{1}{\sigma}} C_{st}^{\frac{\epsilon_j-\sigma}{\sigma}} C_{jt}^{\frac{\sigma-1}{\sigma}}\,. \tag{3.2.4}$$

In (3.2.3), the parameters ξ and ν control the share of expenditure on service goods and the curvature of the income expansion path, respectively. On the other hand, in (3.2.4) the following parametric restrictions are sufficient to guarantee that the aggregator C_{st} is globally monotonically increasing and quasi-concave, yielding a well-defined utility function over the bundle of goods (C_{ht}, C_{mt}): i) $\sigma > 0$ and $\sigma \neq 1$, ii) $\Omega_j > 0$ for all j, and iii) $(\sigma - \epsilon_i)/(\sigma - 1) > 0$ for all $j \in \{h, m\}$ (see Hanoch (1975), pp.403 and Comin, Lashkari and Mestieri (2015), pp. 6).

I assume that higher and lower skilled individuals have the same form of preferences as given in (3.2.2)-(3.2.4) but they differ in terms of the preference parameters:

$$\Phi_{1i}, \Phi_{2i}, \theta_i, \xi_i, \nu_i, \sigma_i, \epsilon_i, \Omega_i \qquad i \in \{h, \ell\}$$

That is, the preferences of an agent depends on whether she is employed as a high skill or low skill worker. This is a critical assumption in the model and in the next section, I present some empirical support for it and provide an extended discussion on its implications in the model.

3.3 Cross-sectional Consumption Patterns

In this section, first I analyze the data obtained from Consumer Expenditure Survey (CES) in order to investigate consumption patterns across different income groups. In their CES, Bureau of Labor Statistics collects consumption expenditures on various categories and report them together with some socio-economic characteristics of the respondents, one of which is the quintile of income that the respondent belongs to. On the other hand, Taylor and Houthakker (2009) classify each consumption category surveyed in CES into necessity,

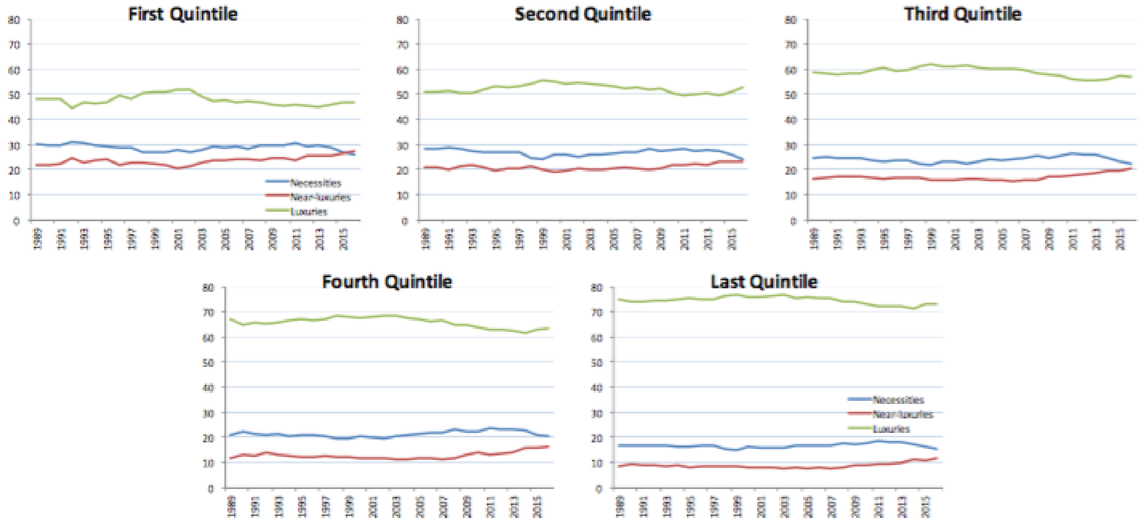

Figure 3: The Shares of Necessities, Near-Luxuries and Luxuries by Income Quintiles, 1989-2016. Source: Bureau of Labor Statistics, CES.

near-luxury or luxury based on their expenditure elasticities.[10] Using the data from CES of Bureau of Labor Statistics and the classification in Taylor and Houthakker (2009), I look at how the total share of each category has evolved in the last 30 years. The results are presented in Figure 3. There are three important conclusions from this exercise:

i) There are stark differences among the income quintiles in terms of the share of spending on luxuries out of the total spending. For instance, the highest income quintile spends approximately 75% on luxuries, whereas the lowest income quintile makes less than 50% of their total expenditures on luxuries.

ii) The share of luxuries in total expenditures decreases steadily from the highest income quintile to the lowest.

iii) These patterns have been approximately stable over the last three decades.

On the other hand, in their survey paper Chai, Rohde and Silber (2015) list three *stylized* facts on the consumption behavior of different income groups:

[10]See Appendix 2 for the details.

Stylized Fact 1: At low income levels, spending diversity is low as food expenditure dominates spending,

Stylized Fact 2: As household income grows, spending diversity increases via reductions in the budget share of food spending and increases in non-food expenditure,

Stylized Fact 3: Individual household spending becomes more diversified as income rises.

The stylized facts reported by Chai, Rohde and Silber (2015) and my findings based on CES data are parallel, except the last stylized fact which cannot be inferred directly from my exercise. Note that the previous analysis implies that income sensitivity of consumption behavior increases as income rises. In accordance with this implication, De Giorgi and Gambetti (2017) find that the right tail of the consumption distribution, comprised mostly of highly educated individuals, has a larger and quicker response than other parts of the distribution to shocks that drive cyclical fluctuations.

Based on these two sets of findings I claim that the preference parameters should vary across the income groups in order to reflect these different consumption patterns and this is what I assumed in the previous section. In particular, note that under the specification of utility function employed in this paper (see equation (3.2.3)-(3.2.4)), these observations can be mapped to the parameters of the utility function; the parameter ξ controls the share of expenditures on goods and ν how fast the share of services increases as the income rises; similarly parameter Ω_h controls the share of home produced services in steady state and parameters ϵ_h and ϵ_m control how fast home and market produced services expand as the income rises.

However, this cross-sectional variability of consumption bundles should not to be confused with *preference heterogeneity* because it is a *preference pattern* that arises mostly due to the income and wealth inequality. In order to develop some further motivation for the assumption I make in this paper, let us use the same consumption aggregator is of a Cobb-Douglas type

$$C_t \equiv C_{gt}^{\gamma} C_{st}^{1-\gamma} .$$

Now, instead of considering γ as a constant suppose that it is a function of the real income

$$\gamma = \gamma(y)\,.$$

With this specification, an agent with real income y spends $\gamma(y)$ fraction of her total consumption expenditures on goods, and $1-\gamma(y)$ on services. More importantly this assumption also entails that agents just adopt the consumption behavior of higher income agents as their income increases, which in return gives rise to a cross-sectional preference pattern that agents approximately move along as their income rise. [11]

As an example, consider the following functional form

$$\gamma(y) = ae^{-by}, \qquad a \in (0,1), \quad y \in [0,\infty), \tag{3.3.1}$$

where the lowest income level is normalized to 0. According to this invariant population preference distribution, the expenditure share of goods is a for the lowest income agent, and $ae^{-b\overline{y}}$ for the highest income agent as shown in the figure below. If the level of income of the highest type rises, say from $\overline{y}$ to $\overline{y}'$, then the expenditure share of goods decreases from $ae^{-b\overline{y}}$ to $ae^{-b\overline{y}'}$. Similarly, other agents move towards the right depending on their new income levels. Using a time-invariant preference structure simplifies the analysis in the following sense: since the agents simply move along a time-invariant curve, we only need to keep track of the measure of agents who push the current frontier of the consumption distribution further and the measure of agents who moved forward from the bottom part of the preference distribution. The difference between the measures of these two groups gives a good idea about the total change in the factor content of goods and services consumed as the economy transits from one point to another.[12]

Finally, in terms of general equilibrium thinking, assuming that agents move along a time-invariant consumption pattern should cause no trouble since in a business cycle model, the main concern should be to replicate the main aggregate statistics and eventually provide some predictions for the course of aggregate variables, not to track the exact consumption bundles agents choose over the business cycle. On the contrary, I claim that imposing

[11] Alternatively, the population preference distribution can be modeled as a function of agents' rank in terms of their income.

[12] Methodologically the story outlined in this paragraph can be considered as a modeling strategy to incorporate a significant amount of heterogeneity in terms of the consumption behavior of agents with different income levels into the model without rendering it intractable.

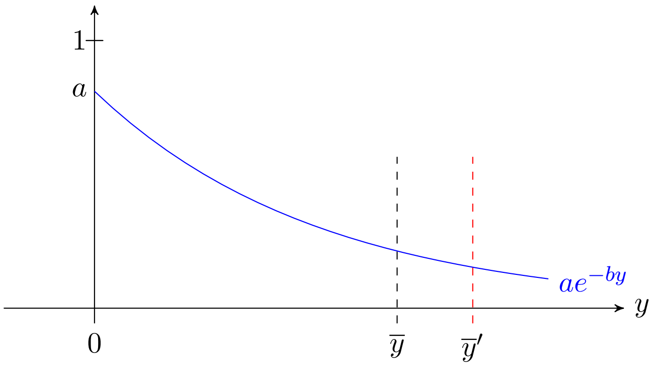

Figure 4: Cross-sectional Consumption Pattern

an empirically plausible discipline on the consumption side in multi-sector business cycle models can only enhance their performance.

4 Model

4.1 Households' Problem

I assume that there is a continuum of identical families of measure one and each family consists of a measure m of high-skill agents and a measure $1-m$ of low-skill agents. Furthermore, I assume that the head of the household acts like a social planner in each family and maximizes the total utility of the family members by choosing infinite sequences of consumption and labor supply for each type and managing the capital stock of the economy. Accordingly, let us denote the total utility of the family members by

$$U(\cdot) = U_h(C_{hgt}, C_{hht}, C_{hmt}, N_{ht}) + U_\ell(C_{\ell gt}, C_{\ell ht}, C_{\ell mt}, N_{\ell t})\,.$$

where the utility function of the high skill agents is the following[13]

$$U_h(C_{hgt}, C_{hst}, N_{ht}) = \frac{\Phi_{1h}}{1-\gamma}\left(C_{ht} - \Phi_{2h}\frac{N_{ht}^{1+\theta_h}}{1+\theta_h}\right)^{1-\gamma}, \qquad 0<\gamma,\, 0<\theta_h$$

[13] Since the describtion of the problem is exactly the same for high and low skill agents I will stick to the problem of high skill agents to avoid additional notation.

N_{ht} denotes the labor supply of the agent; C_{ht} is the consumption aggregate that is defined through the following relations

$$C_{ht} = \min\left\{C_{hst},\ \xi_h C_{hgt}^{\nu_h}\right\}$$
$$1 = \sum_{j\in\{h,m\}}^{I} \Omega_{hj}^{\frac{1}{\sigma_h}} C_{hst}^{\frac{\epsilon_{hj}-\sigma_h}{\sigma_h}} C_{hjt}^{\frac{\sigma_h-1}{\sigma_h}}\ .$$

where C_{hgt}, C_{hht} and C_{hmt} are consumption of basic goods[14], home produced and market provided services, respectively.

Now, we can write the problem that the head of the households solves as follows

$$V(K_t, q_t) = \max_{\{C_{gt}, C_{ht}, C_{mt}, N_{\ell t}, N_{ht}, I_t, u_t\}} \left\{ U(\cdot) +\ \beta\,\mathbb{E}\,V(K_{t+1}, q_{t+1}) \right\}$$

subject to

$$(i)\ \sum_{j\in\{g,h,m\}} P_{jt}C_{jt} + I_t = R_t u_t K_t + w_{\ell t}N_{\ell t} + w_{ht}N_{ht}$$
$$(ii)\ K_{t+1} = (1-\delta(u_t))K_t + I_t q_t\ ,$$
$$(iii)\ C_{jt} = C_{hjt} + C_{\ell jt},\ j \in \{g,h,m\},$$
$$(iv)\ C_{hgt}, C_{hst}, C_{\ell gt}, C_{\ell st} \geq 0\ ,$$
$$(v)\ \text{given } K_0\ .$$

where P_{jt} is the price good j, $j \in \{g,h,m\}$; I_t the total investment; R_t is the rental price of capital services; K_t is the aggregate capital stock; q_t is a stochastic term and w_{ht} ($w_{\ell t}$) is the wage rate for high-skill (low-skill) workers. In the model, all nominal variables are expressed in terms of the investment good.

In words, at the beginning of period t the family starts with K_t units of capital stock inherited from the previous period, and makes four decisions: (i) how much capital services to generate (u_tK_t) in the current period by choosing the capacity utilization rate u_t, (ii) how much new capital (I_t) add to the current capital stock for a given efficiency level q_t, (iii)

[14] I use the terms *basic goods* and *goods* interchangeably in the paper unless there is a risk of confusion.

how much labor to supply ($N_{\ell t}$ and N_{ht}), and (iv) how to allocate the total consumption expenditure among different goods (C_{gt}, C_{ht} and C_{mt}).

Also, as in GHH, the capital utilization decision involves Keynes' notion of *user cost*. That is, a higher utilization rate causes a faster depreciation of the capital stock, either because wear and tear increase with the use or because less time can be devoted to maintenance. This effect is modeled as a variable depreciation cost $\delta(u_t)$ in the capital accumulation equation above. The non-negative depreciation function δ satisfies $0 \leq \delta \leq 1$, $\delta' > 0$, $\delta'' > 0$. In particular, I assume $\delta(u_t) = bu_t^{\tau}/\tau$. The variable I_t is a component of the gross investment, as corresponding to the national income accounts. Its contribution to the production capacity in $t+1$, however, depends on the technological shift factor q_t, affecting the productivity of the new capital goods. The productivity of the already installed capital stock K_t, is not directly affected by the new technology. Correspondingly, K_{t+1} is a measure of the future capital stock in productivity units.

4.2 Firms' Problem

On the production side, there are three representative firms producing basic goods (Y_{gt}), market services (Y_{mt}) and investment goods (Y_{It}), respectively. I assume that there is a complete specialization in the economy such that high skill individuals produce basic goods and low skill individuals produce market services, otherwise each firm solves the usual static problems so I outline their problems only briefly below.

Basic Goods Production. The representative firm that produces (basic) goods, Y_{gt}, employs high-skill labor (H_t) and capital (K_{gt}) in production, and solves:

$$\max_{K_{gt}, H_t} P_{gt} A_{gt} K_{gt}^{1-\alpha_g} H_t^{\alpha_g} - R_t K_{gt} - w_{ht} H_t \,,$$

where K_{gt} is the capital services rented by the goods producing firm.

Market Produced Service Goods. On the other hand, only low-skill labor (L_t) and capital (K_{mt}) are required to produce market produced service goods[15],C_m, so the

[15]This approximately corresponds to *service occupations*. For example, babysitting, dining out or recreational services can be considered as examples of C_m. However, in principle C_m might include some other input combinations as well. For example, any kind of non-compulsory educational service should be considered as a market service in this categorization, and be counted under C_m.

representative firm that produces service goods in the market, Y_{st}, solves:

$$\max_{K_{mt},L_t} P_{mt}\, A_{mt} K_{mt}^{1-\alpha_m} L_t^{\alpha_m} - R_t K_{mt} - w_{\ell t} L_t \,.$$

Home Produced Service Goods. I assume that there is a linear technology accessible to individuals of type $j \in \{h, \ell\}$ to produce services at home, C_{jht}, from basic goods:

$$C_{jht} = \eta_j C_{jgt} \,,$$

where η_j is the productivity of type j in transforming basic goods into service goods at home.

Investment Goods Production. I assume that there is a separate investment good producing firm which uses inputs from both basic goods (M_{gt}) and market services (M_{mt}) to produce the investment good according to a constant elasticity of substitution production function

$$Y_{It} = A_{It} \left(\omega_g^{\frac{1}{\epsilon}} M_{gt}^{\frac{\epsilon-1}{\epsilon}} + \omega_m^{\frac{1}{\epsilon}} M_{mt}^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{1-\epsilon}}$$

where $\varepsilon > 0$ is the elasticity of substitution across input categories and ω_k, $k \in \{g, m\}$, is the relative weight of each input ($\omega_g + \omega_m = 1$).

This specification allows for different degrees of complementarity: if $\varepsilon < 1$, inputs are gross complements, and if $\varepsilon > 1$, inputs are gross substitutes. In the two extreme cases: Y_{It} is Cobb-Douglas, $Y_{It} = M_{gt}^{\omega_g} M_{mt}^{\omega_m}$, for $\varepsilon = 1$, and Y_{It} is Leontief, $Y_{It} = A_{It} \min\{M_{gt}/\omega_g,\, M_{mt}/\omega_m\}$, for $\varepsilon = 0$.

The representative firm producing the investment good solves:

$$\max_{M_{gt},M_{mt}} A_{It} \left(\omega_g^{\frac{1}{\epsilon}} M_{gt}^{\frac{\epsilon-1}{\epsilon}} + \omega_m^{\frac{1}{\epsilon}} M_{mt}^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{1-\epsilon}} - P_{gt} M_{gt} - P_{mt} M_{mt}$$

4.3 Equilibrium

An equilibrium for this economy can be defined as follows: households and firms solve their respective problems as stated in the previous sections and market clearing conditions hold in each market:

- The market clearing conditions for basic goods and services

$$
\begin{aligned}
Y_{gt} &= M_{gt} + C_{gt} \\
Y_{mt} &= M_{mt} + C_{mt} \,.
\end{aligned}
$$

  where

$$
\begin{aligned}
C_{gt} &= C_{hgt} + C_{\ell gt} + \eta_h^{-1} C_{hht} + \eta_\ell^{-1} C_{\ell ht} \\
C_{mt} &= C_{hmt} + C_{\ell mt} \,.
\end{aligned}
$$

  are the quantities that go to consumption out of total production of basic goods and market produced services, respectively.

- Market clearing conditions for labor:

$$
\begin{aligned}
H_t &= N_{ht} \\
L_t &= N_{\ell t} \,.
\end{aligned}
$$

- Markets clearing conditions for capital and investment:

$$
\begin{aligned}
u_t K_t &= K_{gt} + K_{mt} \\
I_t &= Y_{It}
\end{aligned}
$$

  where $u_t K_t$ is the total capital services supplied by the household.

- The law of motion for the capital stock:

$$
K_t = (1 - \delta(u_t)) K_t + I_t q_t \,.
$$

Finally, I define the GDP as the total gross value of consumption and investment expressed in terms of the investment good:

$$
Y_t = P_{gt} \left(C_{hgt} + C_{\ell gt} + \eta_h^{-1} C_{hht} + \eta_\ell^{-1} C_{\ell ht} \right) + P_{mt} \left(C_{hmt} + C_{\ell mt} \right) + I_t \,.
$$

5 A Quantitative Assesement of the Model

In this section, I first characterize the steady state of the economy and then run a series of experiments to assess the performance of the model under different scenarios.

5.1 Steady State

In Appendix 3, I show that the following set of equations characterizes equilibrium of the economy

$$
\begin{aligned}
-\Phi_{1h}\mathcal{U}_{ht}^{-\gamma}+\frac{\xi_h^{-\frac{1}{\nu_h}}}{\nu_h}\mu_{1t}C_{ht}^{\frac{1}{\nu_h}-1} &= \eta_h^{-1}\left(\frac{\epsilon_{hh}-\sigma_h}{\sigma_h-1}\right)\left(\frac{C_{hht}}{C_{ht}}\right)\mu_{1t}+\left(\frac{\epsilon_{hm}-\sigma_h}{\sigma_h-1}\right)\left(\frac{C_{hmt}}{C_{ht}}\right)\mu_{2t}\\
\frac{\eta_h^{-1}\mu_{1t}}{\mu_{2ht}} &= \left(\frac{\Omega_{hh}}{\Omega_{hm}}\right)^{\frac{1}{\sigma_h}} C_{ht}^{\frac{\epsilon_{hh}-\epsilon_{hm}}{\sigma_h}}\left(\frac{C_{hht}}{C_{hmt}}\right)^{-\frac{1}{\sigma_h}}\\
-\Phi_{1\ell}\mathcal{U}_{\ell t}^{-\gamma}+\frac{\xi_\ell^{-\frac{1}{\nu_\ell}}}{\nu_\ell}\mu_{1t}C_{\ell t}^{\frac{1}{\nu_\ell}-1} &= \eta_\ell\left(\frac{\epsilon_{\ell h}-\sigma_\ell}{\sigma_\ell-1}\right)\left(\frac{C_{\ell ht}}{C_{\ell t}}\right)\mu_{1t}+\left(\frac{\epsilon_{\ell m}-\sigma_\ell}{\sigma_\ell-1}\right)\left(\frac{C_{\ell mt}}{C_{\ell t}}\right)\mu_{2t}\\
\frac{\eta_\ell^{-1}\mu_{1t}}{\mu_{2\ell t}} &= \left(\frac{\Omega_{\ell h}}{\Omega_{\ell m}}\right)^{\frac{1}{\sigma_\ell}} C_{\ell t}^{\frac{\epsilon_{\ell h}-\epsilon_{\ell m}}{\sigma_\ell}}\left(\frac{C_{\ell ht}}{C_{\ell mt}}\right)^{-\frac{1}{\sigma_\ell}}\\
\Phi_{1h}\Phi_{2h}\mathcal{U}_{ht}^{-\gamma}N_{ht}^{1+\theta_h} &= \mu_{1t}\,\alpha_g\,Y_{gt}\\
\Phi_{1\ell}\Phi_{2\ell}\mathcal{U}_{\ell t}^{-\gamma}N_{\ell t}^{1+\theta_\ell} &= \mu_{2t}\,\alpha A_m\,Y_{mt}\\
\frac{\mu_{1t}}{\mu_{2t}} &= \left[\frac{\omega_g M_{mt}}{\omega_m M_{gt}}\right]^{\frac{1}{\varepsilon}}\\
\frac{\mu_{1t}}{\mu_{2t}} &= \frac{(1-\alpha_m)Y_m K_g}{(1-\alpha_g)Y_g K_m}\\
\frac{u_t^{\tau-1}}{q_t}\left[\frac{M_{mt}}{\omega_m I_t}\right]^{\frac{1}{\varepsilon}}A_{It}^{\frac{1-\varepsilon}{\varepsilon}} &= \frac{(1-\alpha_m)Y_{mt}}{K_{mt}}\\
\mu_{1t}\left(\frac{Y_{gt}K_{gt}^{-1}}{u_t^{\tau-1}}\right) &= \beta\,\mathbb{E}_t\,\mu_{1,t+1}\left(\frac{Y_{g,t+1}K_{g,t+1}^{-1}}{u_{t+1}^{\tau-1}}\right)\left[1+(1-\frac{1}{\tau})u_{t+1}^{\tau}\right]\\
Y_{gt} &= M_{gt}+\xi_h^{-\frac{1}{\nu_h}}C_{ht}^{\frac{1}{\nu_h}}+\eta_h^{-1}C_{hht}+\xi_\ell^{-\frac{1}{\nu_\ell}}C_{\ell t}^{\frac{1}{\nu_\ell}}+\eta_\ell^{-1}C_{\ell ht}\\
Y_{mt} &= M_{mt}+C_{hmt}+C_{\ell mt}\\
I_t &= \frac{K_{t+1}}{q_t}-[1-\delta(u_t)]\frac{K_t}{q_t}
\end{aligned}
$$

$$
\begin{aligned}
u_t K_t &= K_{gt} + K_{mt} \\
Y_{gt} &= A_{gt} K_{gt}^{1-\alpha_g} N_{ht}^{\alpha_g} \\
Y_{mt} &= A_{mt} K_{mt}^{1-\alpha_m} N_{\ell t}^{\alpha_m} \\
I_t &= A_{It} \left(\omega_g^{\frac{1}{\epsilon}} M_{gt}^{\frac{\epsilon-1}{\epsilon}} + \omega_m^{\frac{1}{\epsilon}} M_{mt}^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}} \\
1 &= \sum_{i \in \{h,m\}} \Omega_{hi}^{\frac{1}{\sigma_h}} C_{ht}^{\frac{\epsilon_{hi}-\sigma_h}{\sigma_h}} C_{hit}^{\frac{\sigma_h-1}{\sigma_h}} \\
1 &= \sum_{i \in \{h,m\}} \Omega_{\ell i}^{\frac{1}{\sigma_\ell}} C_{\ell t}^{\frac{\epsilon_{\ell i}-\sigma_\ell}{\sigma_\ell}} C_{\ell it}^{\frac{\sigma_\ell-1}{\sigma_\ell}}
\end{aligned}
$$

In the system, there are 19 equations in 19 variables:

$$C_h, C_{hh}, C_{hm}, C_\ell, C_{\ell h}, C_{\ell m}, N_h, N_\ell, u, M_g, M_s, K_g, K_m, K, Y_g, Y_m, I, \mu_1, \mu_2$$

where, for convenience, I also define $\mathcal{U}_{jt} \equiv C_{jt} - (1/\theta_j) N^{1+\theta_j}$, $j \in \{h, \ell\}$.

5.2 Experiments

Since my model extends the standard RBC model in a number of dimensions, I try to assess the contribution of these aspects by running a series different scenarios in this section.

First, I assume that the investment shock q_t follows an AR(1) process

$$q_t = \rho q_{t-1} + \nu_t \,, \qquad \nu_t \backsim N(0, \sigma_I^2) \,.$$

In order to choose the parameter values of the stochastic process I follow the same procedure as in GHH. Accordingly I set the magnitude of the innovation and the persistence of the stochastic process as $\sigma_I = 0.045$ and $\rho = 0.52$, respectfully. I also borrow the parameter values for τ, θ_h, σ and β from GHH, and set the fraction of high skill agents to one half, throughout the paper. Finally, I assume that the basic goods constitute 70% of the investment good in the steady state and the elasticity of substitution between basic goods and market produced services is 0.1. I report other parameter values together with the results for each scenario below.

Scenario 1: Homothetic Preferences and Equal Capital Intensities $\approx$ GHH

In order to understand the role of non-homothetic preferences in a multi-sector model I start with a scenario that is approximately equal to a standard one sector model with GHH preferences. For this scenario I assume that the capital intensities across the sectors are the same ($\alpha_g = \alpha_m$) and agents have homothetic preferences with the same preference parameters. The set of parameter values chosen to represent this scenario is given in Table 1 together with the corresponding key ratios realized in the steady state.

γ	2.00	θ_h	0.60	θ_ℓ	0.60	C/Y	0.80
β	0.96	Φ_{h1}	1.00	$\Phi_{\ell 1}$	1.00	I/Y	0.20
A_g	2.50	Φ_{h2}	1.00	$\Phi_{\ell 2}$	1.00	K/Y	2.06
A_s	2.50	Ω_h	0.50	Ω_ℓ	0.50	K_g/uK	0.76
α_g	0.71	σ_h	1.40	σ_ℓ	1.40	Y_g/Y	0.76
α_m	0.71	ϵ_{hh}	1.00	$\epsilon_{\ell h}$	1.00	$P_m Y_m/Y$	0.24
τ	1.42	ϵ_{hm}	1.00	$\epsilon_{\ell m}$	1.00	$\mathcal{U}_h/\mathcal{U}_\ell$	1.00
ω	0.70	ξ_h	1.00	ξ_ℓ	1.00	ρ	0.51
ϵ	0.10	ν_h	1.00	ν_ℓ	1.00	σ	0.045

Table 1: Scenario 1- Paramater Values and Ratios

Table 2 presents the moments of the model for the parameter values given in Table 1. For convenience I also report the corresponding table from GHH together with the US data as reported in their paper. As Table 2 clearly shows, when the capital intensities are the same and the preferences are homothetic, moments of this model is almost identical to those from the model in GHH.[16] Therefore, I take these parameter values as my reference point when I include non-homotheticities and differential capital intensities across the sectors in the next sections.

Scenario 2: Non-homothetic Preferences and Equal Capital Intensities

Now I turn to a more realistic case as my second scenario. The following interpretation is useful to keep in mind as the main motivation behind this scenario and the restrictions imposed on the parameters. In the model, basic goods represent more sophisticated goods and therefore they are produced by high skill agents, and intuitively consumption baskets of high skill agents should contain more of these goods as well. Furthermore, I assume that

[16] Impulse response functions of these two models are also very similar, see Appendix 3

	USA			GHH			Scenario 1		
	Std	AR	Corr	Std	AR	Corr	Std	AR	Corr
Output	3.50	0.66	1.00	3.50	0.66	1.00	3.50	0.66	1.00
Consumption	2.20	0.72	0.74	2.20	0.94	0.79	2.17	0.95	0.80
Investment	10.5	0.25	0.68	11.6	0.50	0.90	11.6	0.50	0.90
Hours	2.10	0.39	0.81	2.20	0.66	1.00	2.21	0.66	1.00
Productivity	2.20	0.77	0.82	1.30	0.66	1.00	1.32	0.66	1.00
Capital	–	–	–	5.60	0.99	0.52	5.72	0.99	0.65
Utilization	–	–	–	6.00	0.52	0.61	6.17	0.53	0.61

Table 2: Scenario 1- Moments

consumption baskets of high skill agents contain more market produced services relative to consumption baskets of low skill agents. With the particular utility form discussed in subsection 3.2 these assumptions translates into the parametric restrictions $\xi_h < \xi_\ell$ and $\Omega_h < \Omega_\ell$, which control the budget shares of basic goods and home produced services, respectively.

In order to motivate the restrictions imposed on the parameters that represent marginal tendencies of buying different goods as income of consumers rises, I continue the story of the previous paragraph as follows. As I discussed in section 1, I assume that agents prioritize increasing the fraction of basic goods in their consumption bundles up to a certain level, and then the propensity to spend on services increases, so the preferences are non-homothetic. Related with the discussion above, I assume that consumption baskets of high-income groups already contain satisfactory amount of basic goods, so they are inclined to buy more service goods ($\nu_h > 1$) and between home produced and market produced service goods they are inclined to buy more market produced service goods ($\epsilon_{hh} < 1 < \epsilon_{hm}$), like time-saving jobs supplied by low skill agents. On the other hand, lower skilled agents prioritize increasing the fraction of basic goods and home produced service goods in their consumption baskets ($\nu_\ell \leq 1$ and $\epsilon_{\ell m} \leq 1 \leq \epsilon_{\ell h}$). This completes the discussion of the restrictions imposed on the parameters and the particular set of parameter values chosen is given in Table 3, together with the realized steady state ratios.

Comparing the results in Table 2 and Table 4 we see that the standard deviation of output rises to 5.35 from 3.52, and its persistence rises to 0.70 from 0.66 after non-homothetic

γ	2.00	θ_h	0.60	θ_ℓ	0.60	C/Y	0.80
β	0.96	Φ_{h1}	1.00	$\Phi_{\ell 1}$	1.00	I/Y	0.20
A_g	2.50	Φ_{h2}	1.00	$\Phi_{\ell 2}$	1.00	K/Y	2.06
A_s	2.50	Ω_h	0.40	Ω_ℓ	0.70	K_g/uK	0.62
α_g	0.71	σ_h	1.40	σ_ℓ	1.40	Y_g/Y	0.62
α_m	0.71	ϵ_{hh}	0.90	$\epsilon_{\ell h}$	1.00	$P_m Y_m/Y$	0.38
τ	1.42	ϵ_{hm}	1.20	$\epsilon_{\ell m}$	1.00	$\mathcal{U}_h/\mathcal{U}_\ell$	1.81
ω	0.70	ξ_h	0.70	ξ_ℓ	1.00	ρ	0.51
ϵ	0.10	ν_h	2.00	ν_ℓ	1.00	σ	0.045

Table 3: Scenario 2- Paramater Values and Ratios

	USA			GHH			Scenario 2		
	Std	AR	Corr	Std	AR	Corr	Std	AR	Corr
Output	3.50	0.66	1.00	3.50	0.66	1.00	5.35	0.70	1.00
Consumption	2.20	0.72	0.74	2.20	0.94	0.79	4.22	0.83	0.97
Investment	10.5	0.25	0.68	11.6	0.50	0.90	11.06	0.55	0.93
Hours	2.10	0.39	0.81	2.20	0.66	1.00	4.31	0.73	1.00
Productivity	2.20	0.77	0.82	1.30	0.66	1.00	1.07	0.62	0.98
Capital	–	–	–	5.60	0.99	0.52	6.19	0.52	0.62
Utilization	–	–	–	6.00	0.52	0.61	6.92	0.52	0.62

Table 4: Scenario 2- Moments

preferences are introduced. Another notable feature of the model is related to the volatility of investment. Recall that a standard criticism of GHH model is that they lead to unrealistic volatility in investment. However, this problem does not arise in my model; the volatility of investment almost remain the same.[17]

The impulse response functions (IRFs) given in Figure 5 offer a better opportunity to understand the underlying reasons that lead to amplification in the model. In the figure, dashed and solid lines denote IRFs with homothetic (Scenario 1) and non-homothetic preferences (Scenario 2), respectively. The figure suggests the following interpretation: low skill agents would like to consume more basic goods and this is possible because high skill agents feel exactly the same for the service goods, therefore the effect of non-homotheticities on prices cancel each other out and consequently agents are able to increase their immediate consumption. In the figure, this can be seen from how IRFs jump up and start with

[17]Volatility of other variables remain approximately the same as well, except the volatility of consumption that I will discuss in the next section.

significantly higher positive intercepts after non-homotheticities are introduced.

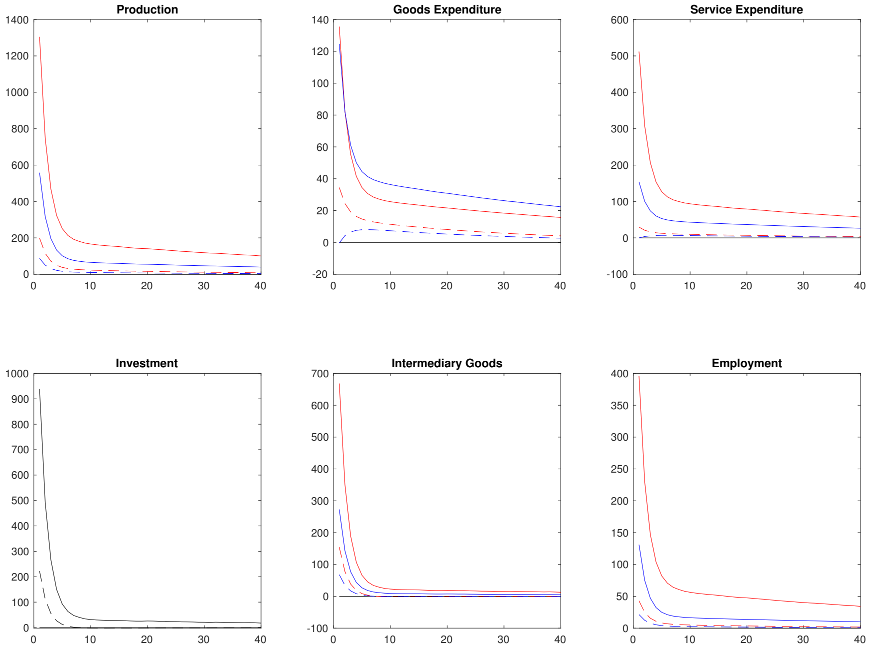

Figure 5: Scenario 2- IRFs

Scenario 3: Non-homothetic Preferences and Different Capital Intensities

In this scenario, I allow the firms producing basic goods and market produced service to operate with different capital intensities. In particular, I assume that high skill agents work in a relatively more capital intensive sector, $\alpha_g < \alpha_m$. This implies that the gross complementarity between capital and labor is higher for high skill jobs.[18] To allow the total hours supplied by low skill agents to vary more as in the data, I also assume that the opportunity cost of working is higher for high skill workers, $\theta_\ell < \theta_h$.

Comparing the results of this scenario with Scenario 2, we see a significant rise in the standard deviation of output from 5.35 to 7.16, and its persistence from 0.70 to 0.72 after

[18] However, note that the opposite of what I assume in this scenario ($\alpha_g > \alpha_m$) can also be justified under the following interpretation: ideas or frontier technologies are the ultimate form of scarcity in the economy and they are reflected in the high skill agents' labor relatively more. As new methods are invented to implement the frontier technologies more cost effectively, the dependence for the high skill labor decreases and they can eventually be performed by relatively lower skilled agents as well. Therefore, capital can be considered as the part of the frontier technology that have been automized by these new methods.

γ	2.00	θ_h	0.60	θ_ℓ	0.30	C/Y	0.80
β	0.96	Φ_{h1}	1.00	$\Phi_{\ell 1}$	1.00	I/Y	0.20
A_g	2.50	Φ_{h2}	1.00	$\Phi_{\ell 2}$	1.00	K/Y	2.06
A_s	2.50	Ω_h	0.40	Ω_ℓ	0.70	K_g/uK	0.62
α_g	0.66	σ_h	1.40	σ_ℓ	1.40	Y_g/Y	0.62
α_m	0.80	ϵ_{hh}	0.90	$\epsilon_{\ell h}$	1.00	$P_m Y_m/Y$	0.38
τ	1.42	ϵ_{hm}	1.20	$\epsilon_{\ell m}$	1.00	$\mathcal{U}_h/\mathcal{U}_\ell$	1.81
ω	0.70	ξ_h	0.70	ξ_ℓ	1.00	ρ	0.51
ϵ	0.10	ν_h	2.00	ν_ℓ	1.00	σ	0.045

Table 5: Scenario 3- Paramater Values and Ratios

	USA			GHH			Scenario 3		
	Std	AR	Corr	Std	AR	Corr	Std	AR	Corr
Output	3.50	0.66	1.00	3.50	0.66	1.00	7.16	0.72	1.00
Consumption	2.20	0.72	0.74	2.20	0.94	0.79	6.14	0.80	0.99
Investment	10.5	0.25	0.68	11.6	0.50	0.90	12.21	0.57	0.94
Hours	2.10	0.39	0.81	2.20	0.66	1.00	6.32	0.74	0.99
Productivity	2.20	0.77	0.82	1.30	0.66	1.00	0.89	0.61	0.97
Capital	–	–	–	5.60	0.99	0.52	6.95	1.00	0.69
Utilization	–	–	–	6.00	0.52	0.61	7.71	0.51	0.64

Table 6: Scenario 3- Moments

differential capital intensities and labor elasticities are allowed.

The mechanics of the model can again be better understood from the sectoral IRFs that are depicted in Figure 6. From the graphs, it is clear that agents increase their consumption very sharply for both goods (compare the solid and the dashed lines), and what makes this possible is the direction that agents want to increase their consumption: each type has a very strong desire to consume the good that the other type produces. In other words, with the particular form of non-homothetic preferences that are assumed in this scenario, agents reciprocate to each other's preferences, which in return leads to a circular interaction between them.

Note that the volatility of investment still remain almost unchanged compare to the reference scenario reported in Table 2. This notable feature of the model is also related to the circular interaction mentioned above, which takes place intra-temporally.

Recall that in standard models, to which our two sector model reduces with symmetric preferences, equal factor intensities and homothetic preferences as assumed in Scenario 1,

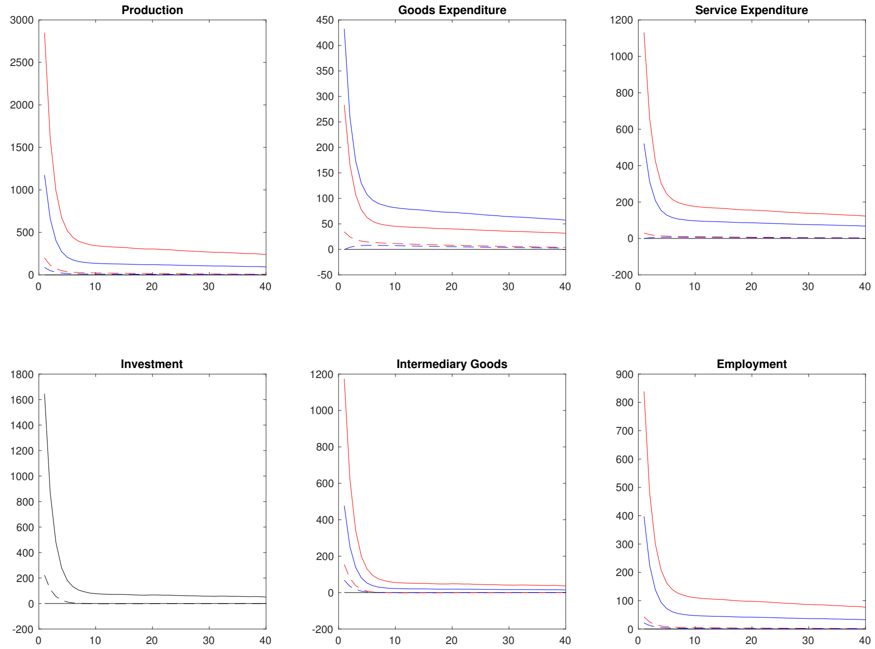

Figure 6: Scenario 3- IRFs

agents prefer to increase their savings to take the advantage of higher interest rates now and start consuming more a little later (dashed lines in the figure corresponds approximately IRFs of a single sector model). But here agents would like to expand their consumptions so badly towards the good that they do not produce, the motive to smooth consumption intertemporally becomes less important in comparison to the opportunity to increase their consumption in the desired direction immediately. But part of this immediate consumption would otherwise turn into the investment good. As a result, the total investment declines and a positive investment shock becomes less effective in boosting the economy. This is the first effect of non-homotheticity that effectively works as if agents have a lower elasticity of intertemporal substitution (EIS) in the model.[19] Consequently, agents invest less and

[19]Also note that when two types of agents have different non-hometheticities, the model effectively works as if there are two types of agents with different EISs. This future has been proven to have important implications for the real business cycle models. For example, Guvenen (2006) shows that an otherwise standard real business cycle model featuring two classes of agents with different EIS is able to produce findings consistent with both capital and consumption fluctuations, and these findings can be reconciled with a low aggregate EIS as long as most of the wealth is held by a small fraction of the population with a high EIS. In these models, EIS of an agent typically depends whether the agent is a saver or borrower. My model provides another mechanism to have differential EIS in business cycle models by showing how

consume more in response to a positive shock, and the model generates more amplification and persistence in investment. On the other hand, the circular interaction between high and low skilled individuals generates counter-balancing effect on the demand for investment goods because high skilled agents can consume more only if they can produce more and since the good they produce is capital intensive this creates demand for investment goods. Our quantitative results show that these two effects balance each other out to a greater extend, which in return prevents excess volatility in investment.

6 Conclusions

In this paper, I studied an economy with two types of agents, heterogenous and non-homothetic preferences and different degrees of specialization. In this environment, through a series of numerical exercises I show that the division of labor together with non-homothetic preferences are highly effective in propagating and amplifying exogenous shocks. I also discuss that the type of the economic environment proposed in this chapter offer an alternative framework to incorporate different features into the business cycles models. One example of this type that I mention in this chapter is the ability of the model to reconcile two different EISs as in saver-borrower models, but with plausible labor market movements.

Although I do not discuss in detail in the paper, the model has no difficulty in generating macroeconomic data that move all together over the business cycle. In particular, the impulse response functions of output, consumption and investment show that what is known as comovement puzzle in the literature does not arise in this model.

One issue that I do not address in this paper is whether the model can produce the right output dynamics, which Benhabib, Perli and Sakellaris (2006) define as hump-shaped impulse response functions and positive autocorrelation coefficients for at least a few periods after the shock. As pointed out by Benhabib, Perli and Sakellaris (2006), the post-impact inverse relationship between consumption and labor of a non-permanent shock is the main reason why the standard RBC models do not produce the right output dynamics. Although my model does not generate hump-shaped impulse response functions for the parameter values chosen in this chapter, this inverse relationship between consumption and labor

non-homothetic preferences and EIS are related.

does not appear in the model, which suggests that including some sorts of labor or capital adjustment costs as in Cogley and Nason (1995) might be an interesting extension to see whether the model can generate hump-shaped output dynamics under these modifications.

Related to the previous point, note that all the action takes place intratemporally which in return restricts the capacity of the model to generate data with higher persistency. This choice obviously was made in order focus on the mechanism that is at the core of the model in an environment that deviates from a standard model as least as possible. However, there are alternative ways to introduce interesting intertemporal dynamics in this framework. In an ongoing project, Karadas (2017), I explore one such possibility. In the model, the preference structure is the same as in the current paper, but on the production side goods producing firms operate under the monopolistic competition and the number of active firms in this sector are determined by an endogenous entry mechanism as in Bilbiie, Ghironi and Melitz (2012). Furthermore, I assume that entering the goods market requires one unit of high-skill labor as a sunk-cost. In the model, the composition of the demand, which varies endogenously just as in the present paper, has an important role: it determines (together with the technology) the number of active firms producing basic goods. In another words, firms need to know demand composition of the economy before deciding to enter, which in return depends on the number of firms entering goods sector. Therefore, an interesting coordination problem arises among entrant firms. In order to investigate the business cycle implications of this coordination problem, I assume that firms receive noisy signals that composed of both their individual demands and the aggregate demand as in Benhabib, Wang and Wen (2015). This framework allows to investigate the contributions of three different factors to the business cycles in a single model: (i) endogenous variation in the demand composition (non-homothetic preferences), (ii) increasing returns to scale (sunk-cost of entering goods market), and (iii) asymmetric information (signal structure).

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Appendix 1

Implicitly Defined Utility Functions

The utility function $v = f(x)$ is *implicitly additive*, if it may be defined by an identity of the form:

$$\sum_k F^k(x_k, v) \equiv 1 \tag{6.0.1}$$

where F^k are n functions of two variables (Hanoch (1975)).[20]

Solving Implicitly Defined Problems

As Hanoch (1975) notes, the problem can be solved either by cost minimizing at given v, or for utility maximization under budget c and prices p. I provide some details below how to proceed in each case.

Expenditure (cost) minimization:

$$\min \sum_k p_k x_k \quad s.t. \quad \sum_k F^k(x_k, v) = 1$$

with FOCs:

$$\lambda F_i^i = p_i, \qquad \forall i$$

Here λ is a Lagrange multiplier, a function of the price vector p and v; it is not marginal cost, since $F_i^i \neq \partial f/\partial x_i = -F_i^i / \sum_k F_v^k$. (Hanock 75, footnote 17)

Utility maximization:

$$\max\ v \quad s.t. \quad \sum_k p_k x_k = c, \quad \sum_k F^k(x_k, v) = 1$$

Set up the Lagrangian[21]

$$\mathcal{L} = v + \rho_1 \left(c - \sum_k p_k x_k \right) + \rho_2 \left(1 - \sum_k F^k(x_k, v) \right)$$

FOCs

$$1 = \rho_2 \sum_k F_v^k$$

$$\rho_1 p_i = -\rho_2 F_i^i$$

[20] It is assumed that F^k are continuously twice differentiable, and satisfy other requirements for a unique, monotone, quasi-concave and closed $f(x)$ for all $x >> 0$ (i.e. for all $x_i > 0$). We rule out in this discussion the linear-isoquant function $\sum f^k(v) \cdot v \equiv 1$, so that $F_{kk} \neq 0$. In principle, the right hand side of (6.0.1) may be a function $h(v)$, but no loss of generality occurs if $h(v)$ is incorporated in F^k.

[21] In Lgrangian formulation we consider $F(x_1, \ldots, x_n, v) = v$ as the objective function and the constraints as two functions g_1 anf g_1 that are again functions of the same $n+1$ variables.

This can be expressed as a single condition

$$p_i = \tilde{\lambda}\, F_i^i,$$

where

$$\tilde{\lambda} = -\frac{1}{\rho_1 \sum_k F_v^k}$$

In Comin, Lashkari and Mestieri (2015), they set up the problem as an utility maximization but they don't work out the first order condition with respect to v because it turns out that because of the specific F^k function they choose they only need the ratio of Lagrange multipliers once the second condition is aggregated over the goods.

Also note that setting up the problem in two different ways clarifies why λ is a function of p and v, as noted by Hanoch (1975).

Appendix 2

In their comprehensive book Taylor and Houthakker (2009) estimates expenditure elasticities of the main consumption categories in Consumption Expenditure Survey (see Table 11.30, pp.205). They also classify these consumption categories according to their total expenditure elasticities as follows: *Necessities:* elasticities that lie between 0 and 0.50; *Near-luxuries:* elasticities that lie between 0.50 and 0.75, *Luxuries:* elasticities that are greater than 0.75. Based on the estimated elasticities and the classification proposed in Taylor and Houthakker (2009), Table 7 presents classification of main consumption categories in CES.

Consumption Category	Elasticity	Classification
Food		
Food at home	0.31	Necessity
Food away from home	0.85	Luxury
Alcoholic beverages	0.47	Necessity
Housing		
Shelter		
Owned dwellings	0.92	Luxury
Rented dwellings	0.58	Near-Luxury
Other lodging	0.85	Luxury
Utilities, fuels, and public services	0.32	Necessity
Household operations	0.92	Luxury
Housekeeping supplies	–	Necessity(*)
Household furnishings and equipment	1.06	Luxury
Apparel and services	0.90	Luxury
Transportation	1.20	
Vehicle purchases (net outlay)	–	Luxury(*)
Gasoline and motor oil	0.39	Necessity
Other vehicle expenses	–	Luxury(*)
Public and other transportation	–	Luxury(*)
Healthcare	0.53	Near-Luxury
Entertainment	0.97	Luxury
Personal care products and services	0.48	Necessity
Reading	0.55	Near-Luxury
Education	0.97	Luxury
Tobacco products and smoking supplies	0.16	Necessity
Miscellaneous	0.57	Near-Luxury
Cash contributions	1.09	Luxury
Personal insurance and pensions	0.97	Luxury

(*) Since the expenditure elasticity of this item is not reported in Taylor and Houthakker (2009) the classification here is based on Henry (2014).

Table 7: Classification of Consumption Categories in CES

Instead of expenditure elasticities, Henry (2014) use the following definitions to classify main consumption categories: *Luxury* is a good or service that is consumed in greater proportions as a person's income increases; *necessity* is a good or service whose consumption is proportionately less as a person's absolute income increases. Interestingly, the classification in Table 7 that is based on Taylor and Houthakker (2009) and the one presented in Henry (2014) are very similar, so the qualitative results discussed in the main text apply to this alternative classification as well.

Appendix 3

The planner's problem with complete specialization is to find a pair of welfare weights $\{\Phi_{1\ell}, \Phi_{1h}\}$ and a sequence of contingency plans $\{C_{jgt}, C_{jht}, C_{jmt}, N_{jt}\}$ for type-$j \in \{h, \ell\}$ consumers and for aggregate allocations $\{K_t, K_{gt}, K_{st}, I_t, M_{gt}, M_{st}, u_t\}$ such that the sum of individual utilities that are weighted by the corresponding welfare weights is maximized:

$$
\begin{aligned}
\max_{\substack{\{C_{it}, C_{ijt}, N_{it}, M_{jt}, K_{jt}, K_{t+1}, u_t\}_{t=0}^{\infty} \\ i\in\{\ell,h\},\, j\in\{g,h,m\}}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ & U_h(C_{ht}, N_{ht}) + U_\ell(C_{\ell t}, N_{\ell t}) \\
& + \mu_{1,t}\left[A_{gt} K_{gt}^{1-\alpha_g} H_t^{\alpha_g} - C_{gt} - M_{gt}\right] \\
& + \mu_{2,t}\left[A_{st} K_{st}^{1-\alpha_m} L_t^{\alpha_m} - C_{st} - M_{st}\right] \\
& + \mu_{3,t}\left[A_{It}\left(\omega_g^{\frac{1}{\epsilon}} M_{gt}^{\frac{\epsilon-1}{\epsilon}} + \omega_m^{\frac{1}{\epsilon}} M_{st}^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}} - I_t\right] \\
& + \mu_{4,t}\left[K_{t+1} - (1-\delta(u_t))\, K_t - I_t q_t\right] \\
& + \mu_{5,t}\left[u K_t - K_{gt} - K_{st}\right] \\
& + \mu_{6,t}\left[C_{gt} - C_{hgt} - C_{\ell gt} - \eta_h^{-1} C_{hht} - \eta_\ell^{-1} C_{\ell ht}\right] \\
& + \mu_{7,t}\left[C_{st} - C_{hmt} - C_{\ell mt}\right] \\
& + \mu_{8,t}\left[H_t - N_{ht}\right] \\
& + \mu_{9,t}\left[L_t - N_{\ell t}\right] \\
& + \mu_{10,t}\left[\sum_{j\in\{h,m\}}^{I} \Omega_{hj}^{\frac{1}{\sigma_h}} C_{hst}^{\frac{\epsilon_{hj}-\sigma_h}{\sigma_h}} C_{hjt}^{\frac{\sigma_h-1}{\sigma_h}} - 1\right] \\
& + \mu_{11,t}\left[\sum_{j\in\{h,m\}}^{I} \Omega_{\ell j}^{\frac{1}{\sigma_\ell}} C_{\ell st}^{\frac{\epsilon_{\ell j}-\sigma_\ell}{\sigma_\ell}} C_{\ell jt}^{\frac{\sigma_\ell-1}{\sigma_\ell}} - 1\right] \Bigg\}
\end{aligned}
$$

where

$$
U_h(C_{hgt}, C_{hht}, C_{hmt}, N_{ht}) = \frac{\Phi_{1h}}{1-\gamma}\left(C_{ht} - \Phi_{2h}\frac{N_{ht}^{1+\theta_h}}{1+\theta_h}\right)^{1-\gamma}
$$

$$
U_\ell(C_{\ell gt}, C_{\ell ht}, C_{\ell mt}, N_{ht}) = \frac{\Phi_{1\ell}}{1-\gamma}\left(C_{\ell t} - \Phi_{2\ell}\frac{N_{\ell t}^{1+\theta_\ell}}{1+\theta_\ell}\right)^{1-\gamma}
$$

and C_h and C_ℓ are implicitly defined by the following relationships

$$
\begin{aligned}
C_{ht} &= \min\left\{C_{hst},\, \xi_h C_{hgt}^{\nu_h}\right\} \\
C_{\ell t} &= \min\left\{C_{\ell st},\, \xi_\ell C_{\ell gt}^{\nu_\ell}\right\} \\
1 &= \sum_{j\in\{h,m\}}^{I} \Omega_{hj}^{\frac{1}{\sigma_h}} C_{hst}^{\frac{\epsilon_{hj}-\sigma_h}{\sigma_h}} C_{hjt}^{\frac{\sigma_h-1}{\sigma_h}} \\
1 &= \sum_{j\in\{h,m\}}^{I} \Omega_{\ell j}^{\frac{1}{\sigma_\ell}} C_{\ell st}^{\frac{\epsilon_{\ell j}-\sigma_\ell}{\sigma_\ell}} C_{\ell jt}^{\frac{\sigma_\ell-1}{\sigma_\ell}}
\end{aligned}
$$

and $\delta(u_t) = (1/\tau)u_t^\tau$, $\tau > 1$.

After a few substitution the problem reduces to the following

$$
\begin{aligned}
\max_{\substack{\{C_{it},C_{ijt},N_{it},M_{jt},K_{jt},K_{t+1},u_t\}_{t=0}^{\infty} \\ i\in\{\ell,h\},\, j\in\{g,h,m\}}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ & U_h(C_{ht}, N_{ht}) + U_\ell(C_{\ell t}, N_{\ell t}) \\
& +\mu_{1,t}\left[A_{gt}K_{gt}^{1-\alpha_g} N_{ht}^{\alpha_g} - M_{gt} - C_{hgt} - \eta_h^{-1} C_{hht} - C_{\ell gt} - \eta_\ell^{-1} C_{\ell ht}\right] \\
& +\mu_{2,t}\left[A_{st}K_{st}^{1-\alpha_m} N_{\ell t}^{\alpha_m} - M_{st} - C_{hmt} - C_{\ell mt}\right] \\
& +\mu_{3,t}\left[\left(\omega_g^{\frac{1}{\epsilon}} M_{gt}^{\frac{\epsilon-1}{\epsilon}} + \omega_m^{\frac{1}{\epsilon}} M_{st}^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}} - \frac{K_{t+1}}{q_t} + [1-\delta(u_t)]\frac{K_t}{q_t}\right] \\
& +\mu_{4,t}\left[u_t K_t - K_{gt} - K_{st}\right] \\
& +\mu_{5,t}\left[\sum_{j\in\{h,m\}}^{I} \Omega_{hj}^{\frac{1}{\sigma_h}} C_{hst}^{\frac{\epsilon_{hj}-\sigma_h}{\sigma_h}} C_{hjt}^{\frac{\sigma_h-1}{\sigma_h}} - 1\right] \\
& +\mu_{6,t}\left[\sum_{j\in\{h,m\}}^{I} \Omega_{\ell j}^{\frac{1}{\sigma_\ell}} C_{\ell st}^{\frac{\epsilon_{\ell j}-\sigma_\ell}{\sigma_\ell}} C_{\ell jt}^{\frac{\sigma_\ell-1}{\sigma_\ell}} - 1\right] \Bigg\}
\end{aligned}
$$

And using the optimality conditions for Leontief preferences

$$
\begin{aligned}
C_{hst} &= C_{ht}, & C_{hgt} &= \xi_h^{-\frac{1}{\nu_h}} C_{ht}^{\frac{1}{\nu_h}} \\
C_{\ell st} &= C_{\ell t}, & C_{\ell gt} &= \xi_\ell^{-\frac{1}{\nu_\ell}} C_{\ell t}^{\frac{1}{\nu_\ell}}
\end{aligned}
$$

we obtain the following simpler problem:

$$\max_{\substack{\{C_{it}, C_{ijt}, N_{it}, M_{jt}, K_{jt}, K_{t+1}, u_t\}_{t=0}^{\infty} \\ i\in\{\ell,h\},\, j\in\{g,h,m\}}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ U_h(C_{ht}, N_{ht}) + U_\ell(C_{\ell t}, N_{\ell t})$$

$$+\mu_{1,t} \left[A_{gt} K_{gt}^{1-\alpha_g} N_{ht}^{\alpha_g} - M_{gt} - \xi_h^{-\frac{1}{\nu_h}} C_{ht}^{\frac{1}{\nu_h}} - \eta_h^{-1} C_{hht} - \xi_\ell^{-\frac{1}{\nu_\ell}} C_{\ell t}^{\frac{1}{\nu_\ell}} - \eta_\ell^{-1} C_{\ell ht} \right]$$

$$+\mu_{2,t} \left[A_{st} K_{st}^{1-\alpha_m} N_{\ell t}^{\alpha_m} - M_{st} - C_{hmt} - C_{\ell mt} \right]$$

$$+\mu_{3,t} \left[\left(\omega_g^{\frac{1}{\epsilon}} M_{gt}^{\frac{\epsilon-1}{\epsilon}} + \omega_m^{\frac{1}{\epsilon}} M_{st}^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}} - \frac{K_{t+1}}{q_t} + [1-\delta(u_t)] \frac{K_t}{q_t} \right]$$

$$+\mu_{4,t} \left[u_t K_t - K_{gt} - K_{st} \right]$$

$$+\mu_{5,t} \left[\sum_{j\in\{h,m\}}^{I} \Omega_{hj}^{\frac{1}{\sigma_h}} C_{ht}^{\frac{\epsilon_{hj}-\sigma_h}{\sigma_h}} C_{hjt}^{\frac{\sigma_h-1}{\sigma_h}} - 1 \right]$$

$$+\mu_{6,t} \left[\sum_{j\in\{h,m\}}^{I} \Omega_{\ell j}^{\frac{1}{\sigma_\ell}} C_{\ell t}^{\frac{\epsilon_{\ell j}-\sigma_\ell}{\sigma_\ell}} C_{\ell jt}^{\frac{\sigma_\ell-1}{\sigma_\ell}} - 1 \right] \Bigg\}$$

The first order conditions (for an interior optimum) are:

$$
\begin{aligned}
C_{ht} \;&: -\Phi_{1h}\mathcal{U}_{ht}^{-\gamma} + \frac{\xi_h^{-\frac{1}{\nu_h}}}{\nu_h}\mu_{1t}C_{ht}^{\frac{1}{\nu_h}-1} = \mu_{5t}\Bigg[\left(\frac{\epsilon_{hh}-\sigma}{\sigma}\right)\Omega_{hh}^{\frac{1}{\sigma}}C_{ht}^{\frac{\epsilon_{hh}-\sigma}{\sigma}-1}C_{hht}^{\frac{\sigma-1}{\sigma}} \\
&\qquad\qquad + \left(\frac{\epsilon_{hm}-\sigma}{\sigma}\right)\Omega_{hm}^{\frac{1}{\sigma}}C_{ht}^{\frac{\epsilon_{hm}-\sigma}{\sigma}-1}C_{hmt}^{\frac{\sigma-1}{\sigma}}\Bigg] \\
C_{hht} \;&: \eta_h^{-1}\mu_{1t} = \mu_{5t}\left(\frac{\sigma-1}{\sigma}\right)\Omega_{hh}^{\frac{1}{\sigma}}C_{ht}^{\frac{\epsilon_{hh}-\sigma}{\sigma}}C_{hht}^{\frac{\sigma-1}{\sigma}-1} \\
C_{hmt} \;&: \mu_{2t} = \mu_{5t}\left(\frac{\sigma-1}{\sigma}\right)\Omega_{hm}^{\frac{1}{\sigma}}C_{ht}^{\frac{\epsilon_{hm}-\sigma}{\sigma}}C_{hmt}^{\frac{\sigma-1}{\sigma}-1} \\
C_{\ell t} \;&: -\Phi_{1\ell}\mathcal{U}_{\ell t}^{-\gamma} + \frac{\xi_\ell^{-\frac{1}{\nu_\ell}}}{\nu_\ell}\mu_{1t}C_{\ell t}^{\frac{1}{\nu_\ell}-1} = \mu_{6t}\Bigg[\left(\frac{\epsilon_{\ell h}-\sigma}{\sigma}\right)\Omega_{\ell h}^{\frac{1}{\sigma}}C_{\ell t}^{\frac{\epsilon_{\ell h}-\sigma}{\sigma}-1}C_{\ell ht}^{\frac{\sigma-1}{\sigma}} \\
&\qquad\qquad + \left(\frac{\epsilon_{\ell m}-\sigma}{\sigma}\right)\Omega_{\ell m}^{\frac{1}{\sigma}}C_{\ell t}^{\frac{\epsilon_{\ell m}-\sigma}{\sigma}-1}C_{\ell mt}^{\frac{\sigma-1}{\sigma}}\Bigg] \\
C_{\ell ht} \;&: \eta_\ell\mu_{1t} = \mu_{6t}\left(\frac{\sigma-1}{\sigma}\right)\Omega_{\ell h}^{\frac{1}{\sigma}}C_{\ell t}^{\frac{\epsilon_{\ell h}-\sigma}{\sigma}}C_{\ell ht}^{\frac{\sigma-1}{\sigma}-1} \\
C_{\ell mt} \;&: \mu_{2t} = \mu_{6t}\left(\frac{\sigma-1}{\sigma}\right)\Omega_{\ell m}^{\frac{1}{\sigma}}C_{\ell t}^{\frac{\epsilon_{\ell m}-\sigma}{\sigma}}C_{\ell mt}^{\frac{\sigma-1}{\sigma}-1} \\
N_{ht} \;&: \Phi_{1h}\Phi_{2h}\mathcal{U}_{ht}^{-\gamma}N_{ht}^{\theta_h} = \mu_{1t}\,\alpha_g\,Y_{gt}\,N_{ht}^{-1} \\
N_{\ell t} \;&: \Phi_{1\ell}\Phi_{2\ell}\mathcal{U}_{\ell t}^{-\gamma}N_{\ell t}^{\theta_\ell} = \mu_{2t}\,\alpha A_m\,Y_{st}\,N_{\ell t}^{-1} \\
M_{gt} \;&: \mu_{1t} = \mu_{3t}A_{It}\left(\omega_g^{\frac{1}{\epsilon}}M_{gt}^{\frac{\epsilon-1}{\epsilon}} + \omega_m^{\frac{1}{\epsilon}}M_{st}^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}-1}\omega_g^{\frac{1}{\epsilon}}M_{gt}^{\frac{\epsilon-1}{\epsilon}-1} \\
M_{st} \;&: \mu_{2t} = \mu_{3t}A_{It}\left(\omega_g^{\frac{1}{\epsilon}}M_{gt}^{\frac{\epsilon-1}{\epsilon}} + \omega_m^{\frac{1}{\epsilon}}M_{st}^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}-1}\omega_m^{\frac{1}{\epsilon}}M_{st}^{\frac{\epsilon-1}{\epsilon}-1} \\
K_{gt} \;&: \mu_{1t}(1-\alpha_g)Y_{gt}K_{gt}^{-1} = \mu_{4t} \\
K_{st} \;&: \mu_{2t}(1-\alpha_m)Y_{st}K_{st}^{-1} = \mu_{4t} \\
u_t \;&: \mu_{3t}\frac{\delta'(u_t)}{q_t} = \mu_{4t} \\
K_{t+1} \;&: \mu_{3t}\frac{1}{q_t} = \beta\,\mathbb{E}_t\left[\mu_{3,t+1}\left(\frac{1-\delta(u_{t+1})}{q_{t+1}}\right) + \mu_{4,t+1}u_{t+1}\right]
\end{aligned}
$$

Combining the last two equations we obtain

$$
\begin{aligned}
\mu_{3t}\frac{1}{q_t} &= \beta\,\mathbb{E}_t\left[\mu_{3,t+1}\left(\frac{1-\delta(u_{t+1})}{q_{t+1}}\right) + \mu_{4,t+1}u_{t+1}\right] \\
\frac{\mu_{4t}}{\delta'(u_t)} &= \beta\,\mathbb{E}_t\left[\frac{\mu_{4,t+1}q_{t+1}}{\delta'(u_{t+1})}\left(\frac{1-\delta(u_{t+1})}{q_{t+1}}\right) + \mu_{4,t+1}u_{t+1}\right] \\
\frac{\mu_{4t}}{u_t^{\tau-1}} &= \beta\,\mathbb{E}_t\,\frac{\mu_{4,t+1}}{u_{t+1}^{\tau-1}}\left[1 + (1-\frac{1}{\tau})u_{t+1}^{\tau}\right] \\
\mu_{1t}\frac{Y_{gt}K_{gt}^{-1}}{u_t^{\tau-1}} &= \beta\,\mathbb{E}_t\,\mu_{1,t+1}\frac{Y_{g,t+1}K_{g,t+1}^{-1}}{u_{t+1}^{\tau-1}}\left[1 + (1-\frac{1}{\tau})u_{t+1}^{\tau}\right]
\end{aligned}
$$

Rearranging the rest of the FOCs, we obtain the following

$$\begin{aligned}
\Phi_{1h}\Phi_{2h}\mathcal{U}_{ht}^{-\gamma}N_{ht}^{1+\theta_h} &= \mu_{1t}\,\alpha_g\,Y_{gt} \\
\Phi_{1\ell}\Phi_{2\ell}\mathcal{U}_{\ell t}^{-\gamma}N_{\ell t}^{1+\theta_\ell} &= \mu_{2t}\,\alpha A_m\,Y_{st} \\
\frac{\mu_{1t}}{\mu_{2t}} &= \left[\frac{\omega_g M_{st}}{\omega_m M_{gt}}\right]^{\frac{1}{\varepsilon}} \\
\mu_{1t}M_{gt} + \mu_{2t}M_{st} &= \mu_{3t}I_t \\
\frac{\mu_{1t}}{\mu_{2t}} &= \frac{(1-\alpha_m)Y_{st}K_{gt}}{(1-\alpha_g)Y_{gt}K_{st}} \\
\mu_{1t}(1-\alpha_g)Y_{gt} + \mu_{2t}(1-\alpha_m)Y_{st} &= \mu_{4t}u_tK_t \\
\mu_{3t}\frac{\delta'(u_t)}{q_t} &= \mu_{4t} \\
\mu_{1t}\frac{Y_{gt}K_{gt}^{-1}}{u_t^{\tau-1}} &= \beta\mathbb{E}_t\mu_{1,t+1}\frac{Y_{g,t+1}K_{g,t+1}^{-1}}{u_{t+1}^{\tau-1}}\left[1+(1-\frac{1}{\tau})u_{t+1}^{\tau}\right]
\end{aligned}$$

Finally, work on the second last equation in the system:

$$\begin{aligned}
\frac{\mu_{3t}}{\mu_{2t}}\frac{\delta'(u_t)}{q_t} &= \frac{\mu_{4t}}{\mu_{2t}} \\
\frac{\delta'(u_t)}{q_t}\frac{1}{I_t}\left(\frac{\mu_{1t}}{\mu_{2t}}M_{gt}+M_{st}\right) &= \frac{1}{u_tK_t}\left(\frac{\mu_{1t}}{\mu_{2t}}(1-\alpha_g)Y_{gt}+(1-\alpha_m)Y_{st}\right) \\
\frac{\delta'(u_t)}{q_t}\frac{1}{I_t}\left(\left[\frac{\omega_g M_{st}}{\omega_m M_{gt}}\right]^{\frac{1}{\varepsilon}}M_{gt}+M_{st}\right) &= \frac{1}{u_tK_t}\left(\frac{(1-\alpha_m)Y_{st}K_{gt}}{(1-\alpha_g)Y_{gt}K_{st}}(1-\alpha_g)Y_{gt}+(1-\alpha_m)Y_{st}\right) \\
\frac{\delta'(u_t)}{q_t}\frac{1}{I_t}\left[\frac{M_{st}}{\omega_m}\right]^{\frac{1}{\varepsilon}}\left[\frac{I_t}{A_{It}}\right]^{\frac{\varepsilon-1}{\varepsilon}} &= \frac{1}{u_tK_t}\left(\frac{(1-\alpha_m)Y_{st}K_{gt}}{K_{st}}+(1-\alpha_m)Y_{st}\right) \\
\frac{u_t^{\tau-1}}{q_t}\left[\frac{M_{st}}{\omega_m I_t}\right]^{\frac{1}{\varepsilon}}A_{It}^{\frac{1-\varepsilon}{\varepsilon}} &= \frac{(1-\alpha_m)Y_{st}}{K_{st}}
\end{aligned}$$

The complete system (together with definitions) reads:

$$
\begin{aligned}
-\Phi_{1h}\mathcal{U}_{ht}^{-\gamma} + \frac{\xi_h^{-\frac{1}{\nu_h}}}{\nu_h}\mu_{1t}C_{ht}^{\frac{1}{\nu_h}-1} &= \eta_h^{-1}\left(\frac{\epsilon_{hh}-\sigma_h}{\sigma_h-1}\right)\left(\frac{C_{hht}}{C_{ht}}\right)\mu_{1t} + \left(\frac{\epsilon_{hm}-\sigma_h}{\sigma_h-1}\right)\left(\frac{C_{hmt}}{C_{ht}}\right)\mu_{2t} \\
\frac{\eta_h^{-1}\mu_{1t}}{\mu_{2ht}} &= \left(\frac{\Omega_{hh}}{\Omega_{hm}}\right)^{\frac{1}{\sigma_h}} C_{ht}^{\frac{\epsilon_{hh}-\epsilon_{hm}}{\sigma_h}}\left(\frac{C_{hht}}{C_{hmt}}\right)^{-\frac{1}{\sigma_h}} \\
-\Phi_{1\ell}\mathcal{U}_{\ell t}^{-\gamma} + \frac{\xi_\ell^{-\frac{1}{\nu_\ell}}}{\nu_\ell}\mu_{1t}C_{\ell t}^{\frac{1}{\nu_\ell}-1} &= \eta_\ell^{-1}\left(\frac{\epsilon_{\ell h}-\sigma_\ell}{\sigma_\ell-1}\right)\left(\frac{C_{\ell ht}}{C_{\ell t}}\right)\mu_{1t} + \left(\frac{\epsilon_{\ell m}-\sigma_\ell}{\sigma_\ell-1}\right)\left(\frac{C_{\ell mt}}{C_{\ell t}}\right)\mu_{2t} \\
\frac{\eta_\ell^{-1}\mu_{1t}}{\mu_{2\ell t}} &= \left(\frac{\Omega_{\ell h}}{\Omega_{\ell m}}\right)^{\frac{1}{\sigma_\ell}} C_{\ell t}^{\frac{\epsilon_{\ell h}-\epsilon_{\ell m}}{\sigma_\ell}}\left(\frac{C_{\ell ht}}{C_{\ell mt}}\right)^{-\frac{1}{\sigma_\ell}} \\
\Phi_{1h}\Phi_{2h}\mathcal{U}_{ht}^{-\gamma}N_{ht}^{1+\theta_h} &= \mu_{1t}\,\alpha_g\,Y_{gt} \\
\Phi_{1\ell}\Phi_{2\ell}\mathcal{U}_{\ell t}^{-\gamma}N_{\ell t}^{1+\theta_\ell} &= \mu_{2t}\,\alpha A_m\,Y_{st} \\
\frac{\mu_{1t}}{\mu_{2t}} &= \left[\frac{\omega_g M_{st}}{\omega_m M_{gt}}\right]^{\frac{1}{\varepsilon}} \\
\frac{\mu_{1t}}{\mu_{2t}} &= \frac{(1-\alpha_m)Y_m K_g}{(1-\alpha_g)Y_g K_s} \\
\frac{u_t^{\tau-1}}{q_t}\left[\frac{M_{st}}{\omega_m I_t}\right]^{\frac{1}{\varepsilon}} A_{It}^{\frac{1-\varepsilon}{\varepsilon}} &= \frac{(1-\alpha_m)Y_{st}}{K_{st}} \\
\mu_{1t}\left(\frac{Y_{gt}K_{gt}^{-1}}{u_t^{\tau-1}}\right) &= \beta\,\mathbb{E}_t\,\mu_{1,t+1}\left(\frac{Y_{g,t+1}K_{g,t+1}^{-1}}{u_{t+1}^{\tau-1}}\right)\left[1+(1-\frac{1}{\tau})u_{t+1}^{\tau}\right] \\
Y_{gt} &= M_{gt} + \xi_h^{-\frac{1}{\nu_h}}C_{ht}^{\frac{1}{\nu_h}} + \eta_h^{-1}C_{hht} + \xi_\ell^{-\frac{1}{\nu_\ell}}C_{\ell t}^{\frac{1}{\nu_\ell}} + \eta_\ell^{-1}C_{\ell ht} \\
Y_{st} &= M_{st} + C_{hmt} + C_{\ell mt} \\
I_t &= \frac{K_{t+1}}{q_t} - [1-\delta(u_t)]\frac{K_t}{q_t} \\
u_t K_t &= K_{gt} + K_{st} \\
Y_{gt} &= A_{gt}K_{gt}^{1-\alpha_g}N_{ht}^{\alpha_g} \\
Y_{st} &= A_{st}K_{st}^{1-\alpha_m}N_{\ell t}^{\alpha_m} \\
I_t &= A_{It}\left(\omega_g^{\frac{1}{\epsilon}}M_{gt}^{\frac{\epsilon-1}{\epsilon}} + \omega_m^{\frac{1}{\epsilon}}M_{st}^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}} \\
1 &= \sum_{i\in\{g,s\}}\Omega_{hi}^{\frac{1}{\sigma_h}}C_{ht}^{\frac{\epsilon_{hi}-\sigma_h}{\sigma_h}}C_{hit}^{\frac{\sigma_h-1}{\sigma_h}} \\
1 &= \sum_{i\in\{g,s\}}\Omega_{\ell i}^{\frac{1}{\sigma_\ell}}C_{\ell t}^{\frac{\epsilon_{\ell i}-\sigma_\ell}{\sigma_\ell}}C_{\ell it}^{\frac{\sigma_\ell-1}{\sigma_\ell}}
\end{aligned}
$$

There are 19 equations and 19 variables:

$$C_h, C_{hh}, C_{hm}, C_\ell, C_{\ell h}, C_{\ell m}, N_h, N_\ell, u, M_g, M_s, K_g, K_s, K, Y_g, Y_m, I, \mu_1, \mu_2$$

Computing GDP

Recall that I defined GDP as the total gross value of consumption and investment expressed in terms of the investment good:

$$Y_t = P_{gt}\left(C_{hgt} + C_{\ell gt} + \eta_h^{-1} C_{hht} + \eta_\ell^{-1} C_{\ell ht}\right) + P_{mt}\left(C_{hmt} + C_{\ell mt}\right) + I_t\,.$$

Obviously, in order to be able to compute GDP we need to know prices P_{gt} and P_{st}. But note that the investment producing firm's problem contains both prices that we need:

$$\max_{M_{gt}, M_{mt}} A_{It}\left(\omega_g^{\frac{1}{\epsilon}} M_{gt}^{\frac{\epsilon-1}{\epsilon}} + \omega_m^{\frac{1}{\epsilon}} M_{mt}^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{1-\epsilon}} - P_{gt} M_{gt} - P_{mt} M_{mt}$$

First order conditions of this problem determines the demand for each good by the investment firm:

$$M_{gt} = \frac{\omega_g P_g^{-\epsilon}}{\omega_g P_g^{1-\epsilon} + \omega_m P_m^{1-\epsilon}} Y_{It}$$
$$M_{mt} = \frac{\omega_m P_m^{-\epsilon}}{\omega_g P_g^{1-\epsilon} + \omega_m P_m^{1-\epsilon}} Y_{It}$$

Substituting the expressions for M_{gt} and M_{mt} into Y_{It} gives

$$1 = A_{It}^{\epsilon-1}[\omega_g P_{gt}^{1-\epsilon} + \omega_m P_{mt}^{1-\epsilon}]$$

which in return allows us to rewrite the factor demand functions simply as

$$M_{gt} = A_{It}^{\epsilon-1} \omega_g P_{gt}^{-\epsilon} Y_{It}$$
$$M_{mt} = A_{It}^{\epsilon-1} \omega_m P_{mt}^{-\epsilon} Y_{It}\,.$$

From these expressions we obtain the following expressions for prices

$$P_{gt} = A_{It}^{\frac{\epsilon-1}{\epsilon}} \left(\frac{\omega_g I_t}{M_{gt}}\right)^{\frac{1}{\epsilon}}$$
$$P_{mt} = A_{It}^{\frac{\epsilon-1}{\epsilon}} \left(\frac{\omega_m I_t}{M_{mt}}\right)^{\frac{1}{\epsilon}}$$

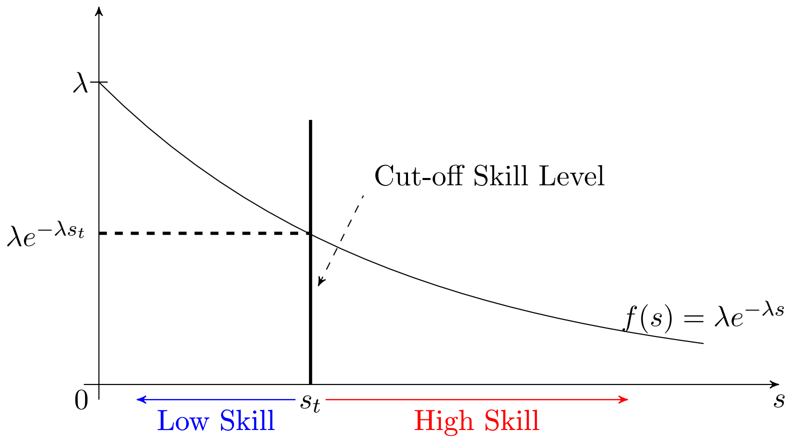

Figure 7: Skill Distribution

Appendix 4: Extension to Incomplete Specialization

The model presented in section 4 assumes that agents are completely specialized in the production of a specific good. In this section I discuss, as a natural extension, how the results change when the specialization is incomplete and therefore agents can switch back and forth between the sectors over the business cycle.[22]

I assume that each household $s \in [0, 1]$ is endowed with a vector of efficiency units as a low and high skill worker, denoted by $e_\ell(s)$ and $e_h(s)$, respectively. Workers have the same efficiency units (normalized to unity) at performing low skill jobs, i.e. $e_\ell(s) = 1, \forall s$. However, agents have heterogeneous skills in performing high skill jobs. Since this is the only heterogeneity among the agents, I will use the same letter to denote the agent's efficiency units in goods production, i.e. s equal the worker's skill in high skill jobs, measured in efficiency units. I assume that skills s are distributed according to a density function $f(s)$ across the population.

For simplicity, I assume that efficiency units (skills) are distributed exponentially on the interval $[0, \infty]$ with the density function $f(s) = \lambda e^{-\lambda s}$. The optimal allocation of labor between two sectors requires that there is a cut-off skill level s_t such that agents with skill below s_t take low skill jobs and the rest work as high skill workers (see Figure 7).

The main mechanism is the same as in the previous section, except the fraction of agents working in goods sector, m_t, varies over the business which affects the persistence of the endogenous variables.

The planner's problem with incomplete specialization is to find a pair of welfare weights $\{\Phi_{1\ell}, \Phi_{1h}\}$ and a sequence of contingency plans $\{C_{jgt}, C_{jht}, C_{jmt}, N_{jt}\}$ for type-$j \in \{h, \ell\}$ consumers and for aggregate allocations $\{K_t, K_{gt}, K_{st}, I_t, M_{gt}, M_{st}, u_t, s_t\}$ such that the sum

[22] The extension presented in section closely follows the model in my paper *The Role of Correlated Investment-Skill Shocks in Business Cycles.*

of individual utilities that are weighted by the corresponding welfare weights is maximized:

$$
\begin{aligned}
\max_{\substack{\{C_{it},C_{ijt},N_{it},M_{jt},K_{jt},K_{t+1},u_t\}_{t=0}^{\infty}\\ i\in\{\ell,h\},\, j\in\{g,h,m\}}} \mathbb{E}_0 \sum_{t=0}^{\infty}\beta^t \Bigg\{ & m_t U_h(C_{ht},N_{ht}) + (1-m_t)U_\ell(C_{\ell t},N_{\ell t}) \\
&+\mu_{1,t}\left[A_{gt}K_{gt}^{1-\alpha_g}N_{ht}^{\alpha_g} - M_{gt} - m_t(C_{hgt}+\eta_h^{-1}C_{hht}) - (1-m_t)(C_{\ell gt}+\eta_\ell^{-1}C_{\ell ht})\right] \\
&+\mu_{2,t}\left[A_{st}K_{st}^{1-\alpha_s}N_{\ell t}^{\alpha_s} - M_{st} - m_t C_{hmt} - (1-m_t)C_{\ell mt}\right] \\
&+\mu_{3,t}\left[\left(\omega_g^{\frac{1}{\epsilon}}M_{gt}^{\frac{\epsilon-1}{\epsilon}} + \omega_s^{\frac{1}{\epsilon}}M_{st}^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}} - \frac{K_{t+1}}{q_t} + [1-\delta(u_t)]\frac{K_t}{q_t}\right] \\
&+\mu_{4,t}\left[u_t K_t - K_{gt} - K_{st}\right] \\
&+\mu_{5,t}\left[\sum_{j\in\{h,m\}}^{I}\Omega_{hj}^{\frac{1}{\sigma_h}}C_{hst}^{\frac{\epsilon_{hj}-\sigma_h}{\sigma_h}}C_{hjt}^{\frac{\sigma_h-1}{\sigma_h}} - 1\right] \\
&+\mu_{6,t}\left[\sum_{j\in\{h,m\}}^{I}\Omega_{\ell j}^{\frac{1}{\sigma_\ell}}C_{\ell st}^{\frac{\epsilon_{\ell j}-\sigma_\ell}{\sigma_\ell}}C_{\ell jt}^{\frac{\sigma_\ell-1}{\sigma_\ell}} - 1\right]\Bigg\}
\end{aligned}
$$

where

$$U_h(C_{hgt},C_{hht},C_{hmt}) = \Phi_{1h}\frac{C_{ht}^{1-\gamma}}{1-\gamma}$$

and C_h is implicitly defined by the following relationships

$$
\begin{aligned}
C_{ht} &= \min\left\{C_{hst},\, \xi_h C_{hgt}^{\nu_h}\right\} \\
C_{\ell t} &= \min\left\{C_{\ell st},\, \xi_\ell C_{\ell gt}^{\nu_\ell}\right\} \\
1 &= \sum_{j\in\{h,m\}}^{I}\Omega_{hj}^{\frac{1}{\sigma_h}}C_{hst}^{\frac{\epsilon_{hj}-\sigma_h}{\sigma_h}}C_{hjt}^{\frac{\sigma_h-1}{\sigma_h}} \\
1 &= \sum_{j\in\{h,m\}}^{I}\Omega_{\ell j}^{\frac{1}{\sigma_\ell}}C_{\ell st}^{\frac{\epsilon_{\ell j}-\sigma_\ell}{\sigma_\ell}}C_{\ell jt}^{\frac{\sigma_\ell-1}{\sigma_\ell}}
\end{aligned}
$$

Optimality requires:

$$
\begin{aligned}
C_{hst} &= C_{ht}, & C_{hgt} &= \xi_h^{-\frac{1}{\nu_h}}C_{ht}^{\frac{1}{\nu_h}} \\
C_{\ell st} &= C_{\ell t}, & C_{\ell gt} &= \xi_\ell^{-\frac{1}{\nu_\ell}}C_{\ell t}^{\frac{1}{\nu_\ell}}
\end{aligned}
$$

After a few substitutions the problem becomes:

$$
\begin{aligned}
\max_{\substack{\{C_{it},C_{ijt},N_{it},M_{jt},K_{jt},K_{t+1},u_t\}_{t=0}^{\infty}\\ i\in\{\ell,h\},\, j\in\{g,h,m\}}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ & m_t U_h(C_{ht}, N_{ht}) + (1-m_t) U_\ell(C_{\ell t}, N_{\ell t}) \\
& + \mu_{1,t} \Bigg[A_{gt} K_{gt}^{1-\alpha_g} N_{ht}^{\alpha_g} - M_{gt} - m_t \left(\xi_h^{-\frac{1}{\nu_h}} C_{ht}^{\frac{1}{\nu_h}} + \eta_h^{-1} C_{hht} \right) \\
& \qquad - (1-m_t) \left(\xi_\ell^{-\frac{1}{\nu_\ell}} C_{\ell t}^{\frac{1}{\nu_\ell}} + \eta_\ell^{-1} C_{\ell ht} \right) \Bigg] \\
& + \mu_{2,t} \left[A_{st} K_{st}^{1-\alpha_s} N_{\ell t}^{\alpha_s} - M_{st} - m_t C_{hmt} - (1-m_t) C_{\ell mt} \right] \\
& + \mu_{3,t} \left[\left(\omega_g^{\frac{1}{\epsilon}} M_{gt}^{\frac{\epsilon-1}{\epsilon}} + \omega_s^{\frac{1}{\epsilon}} M_{st}^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}} - \frac{K_{t+1}}{q_t} + [1-\delta(u_t)] \frac{K_t}{q_t} \right] \\
& + \mu_{4,t} \left[u_t K_t - K_{gt} - K_{st} \right] \\
& + \mu_{5,t} \left[\sum_{j\in\{h,m\}}^{I} \Omega_{hj}^{\frac{1}{\sigma_h}} C_{ht}^{\frac{\epsilon_{hj}-\sigma_h}{\sigma_h}} C_{hjt}^{\frac{\sigma_h-1}{\sigma_h}} - 1 \right] \\
& + \mu_{6,t} \left[\sum_{j\in\{h,m\}}^{I} \Omega_{\ell j}^{\frac{1}{\sigma_\ell}} C_{\ell t}^{\frac{\epsilon_{\ell j}-\sigma_\ell}{\sigma_\ell}} C_{\ell jt}^{\frac{\sigma_\ell-1}{\sigma_\ell}} - 1 \right] \Bigg\}
\end{aligned}
$$

where

$$
\begin{aligned}
Y_{gt} &= A_{gt} K_{gt}^{1-\alpha_g} N_{ht}^{\alpha_g} \\
Y_{st} &= A_{st} K_{st}^{1-\alpha_s} N_{\ell t}^{\alpha_s} \\
N_{ht} &= \chi_h \int_{s_t}^{\infty} e_h(s) f(s)\, ds = \frac{\chi_h}{\lambda}(1+\lambda s_t) e^{-\lambda s_t} \\
N_{\ell t} &= \chi_\ell \int_0^{s_t} f(s)\, ds = \chi_\ell (1 - e^{-\lambda s_t}) \\
m_t &= e^{-\lambda s_t} \\
\delta(u_t) &= (1/\tau) u_t^{\tau}, \quad \tau > 1\,.
\end{aligned}
$$

and $m_t = 1 - F(s_t)$ is the measure of high skill agents and $\delta(u_t) = (1/\tau)u_t^\tau$ is an increasing, convex function with $\tau > 1$.

The first order conditions (for an interior optimum) are:

$$
\begin{aligned}
C_{ht} \quad &: \quad -m_t \Phi_{1h} C_{ht}^{-\gamma} + \frac{\xi_h^{-\frac{1}{\nu_h}}}{\nu_h} \mu_{1t} m_t C_{ht}^{\frac{1}{\nu_h}-1} = \mu_{5t} \Bigg[\left(\frac{\epsilon_{hh} - \sigma}{\sigma} \right) \Omega_{hh}^{\frac{1}{\sigma}} C_{ht}^{\frac{\epsilon_{hh}-\sigma}{\sigma}-1} C_{hht}^{\frac{\sigma-1}{\sigma}} \\
&\qquad + \left(\frac{\epsilon_{hm} - \sigma}{\sigma} \right) \Omega_{hm}^{\frac{1}{\sigma}} C_{ht}^{\frac{\epsilon_{hm}-\sigma}{\sigma}-1} C_{hmt}^{\frac{\sigma-1}{\sigma}} \Bigg] \\
C_{hht} \quad &: \quad \eta_h^{-1} m_t \mu_{1t} = \mu_{5t} \left(\frac{\sigma - 1}{\sigma} \right) \Omega_{hh}^{\frac{1}{\sigma}} C_{ht}^{\frac{\epsilon_{hh}-\sigma}{\sigma}} C_{hht}^{\frac{\sigma-1}{\sigma}-1} \\
C_{hmt} \quad &: \quad m_t \mu_{2t} = \mu_{5t} \left(\frac{\sigma - 1}{\sigma} \right) \Omega_{hm}^{\frac{1}{\sigma}} C_{ht}^{\frac{\epsilon_{hm}-\sigma}{\sigma}} C_{hmt}^{\frac{\sigma-1}{\sigma}-1} \\
C_{\ell t} \quad &: \quad (1-m_t) \Phi_{1\ell} C_{\ell t}^{-\gamma} + \frac{\xi_\ell^{-\frac{1}{\nu_\ell}}}{\nu_\ell} \mu_{1t} (1-m_t) C_{\ell t}^{\frac{1}{\nu_\ell}-1} = \mu_{6t} \Bigg[\left(\frac{\epsilon_{\ell h} - \sigma}{\sigma} \right) \Omega_{\ell h}^{\frac{1}{\sigma}} C_{\ell t}^{\frac{\epsilon_{\ell h}-\sigma}{\sigma}-1} C_{\ell ht}^{\frac{\sigma-1}{\sigma}} \\
&\qquad + \left(\frac{\epsilon_{\ell m} - \sigma}{\sigma} \right) \Omega_{\ell m}^{\frac{1}{\sigma}} C_{\ell t}^{\frac{\epsilon_{\ell m}-\sigma}{\sigma}-1} C_{\ell mt}^{\frac{\sigma-1}{\sigma}} \Bigg] \\
C_{\ell ht} \quad &: \quad \eta_\ell^{-1} (1-m_t) \mu_{1t} = \mu_{6t} \left(\frac{\sigma - 1}{\sigma} \right) \Omega_{\ell h}^{\frac{1}{\sigma}} C_{\ell t}^{\frac{\epsilon_{\ell h}-\sigma}{\sigma}} C_{\ell ht}^{\frac{\sigma-1}{\sigma}-1} \\
C_{\ell mt} \quad &: \quad (1-m_t) \mu_{2t} = \mu_{6t} \left(\frac{\sigma - 1}{\sigma} \right) \Omega_{\ell m}^{\frac{1}{\sigma}} C_{\ell t}^{\frac{\epsilon_{\ell m}-\sigma}{\sigma}} C_{\ell mt}^{\frac{\sigma-1}{\sigma}-1} \\
M_{gt} \quad &: \quad \mu_{1t} = \mu_{3t} A_{It} \left(\omega_g^{\frac{1}{\epsilon}} M_{gt}^{\frac{\epsilon-1}{\epsilon}} + \omega_s^{\frac{1}{\epsilon}} M_{st}^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}-1} \omega_g^{\frac{1}{\epsilon}} M_{gt}^{\frac{\epsilon-1}{\epsilon}-1} \\
M_{st} \quad &: \quad \mu_{2t} = \mu_{3t} A_{It} \left(\omega_g^{\frac{1}{\epsilon}} M_{gt}^{\frac{\epsilon-1}{\epsilon}} + \omega_s^{\frac{1}{\epsilon}} M_{st}^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}-1} \omega_s^{\frac{1}{\epsilon}} M_{st}^{\frac{\epsilon-1}{\epsilon}-1} \\
K_{gt} \quad &: \quad \mu_{1t} (1-\alpha_g) Y_{gt} K_{gt}^{-1} = \mu_{4t} \\
K_{st} \quad &: \quad \mu_{2t} (1-\alpha_s) Y_{st} K_{st}^{-1} = \mu_{4t} \\
u_t \quad &: \quad \mu_{3t} \frac{\delta'(u_t)}{q_t} = \mu_{4t} \\
K_{t+1} \quad &: \quad \mu_{3t} \frac{1}{q_t} = \beta \, \mathbb{E}_t \left[\mu_{3,t+1} \left(\frac{1-\delta(u_{t+1})}{q_{t+1}} \right) + \mu_{4,t+1} u_{t+1} \right]
\end{aligned}
$$

and the optimality condition for the cut-off skill level s_t is

$$
\begin{aligned}
0 =& m_t' U_h(\cdot) - m_t' U_\ell(\cdot) \\
&+ \mu_{1,t} \left[\alpha_g Y_{gt} N_{ht}^{-1} N_{ht}' - m_t' \left(\xi_h^{-\frac{1}{\nu_h}} C_{ht}^{\frac{1}{\nu_h}} + \eta_h^{-1} C_{hht} - \xi_\ell^{-\frac{1}{\nu_\ell}} C_{\ell t}^{\frac{1}{\nu_\ell}} - \eta_\ell^{-1} C_{\ell ht} \right) \right] \\
&+ \mu_{2,t} \left[\alpha_s Y_{st} N_{\ell t}^{-1} N_{\ell t}' - m_t' \left(C_{hmt} - C_{\ell mt} \right) \right],
\end{aligned}
$$

where m_t' denotes dm_t/ds_t, and similarly with the other variables.

$$
\begin{aligned}
0 =& m_t' U_h(\cdot) - m_t' U_\ell(\cdot) \\
&+\mu_{1,t}\left[\alpha_g Y_{gt} N_{ht}^{-1} N_{ht}^{'} - m_t' \left(\xi_h^{-\frac{1}{\nu_h}} C_{ht}^{\frac{1}{\nu_h}} + \eta_h^{-1} C_{hht} - \xi_\ell^{-\frac{1}{\nu_\ell}} C_{\ell t}^{\frac{1}{\nu_\ell}} - \eta_\ell^{-1} C_{\ell ht} \right) \right] \\
&+\mu_{2,t}\left[\alpha_s Y_{st} N_{\ell t}^{-1} N_{\ell t}^{'} - m_t^{'} \left(C_{hmt} - C_{\ell mt} \right) \right] \\
0 =& m_t' U_h(\cdot) + \mu_{1t}\, \alpha_g\, Y_{gt}\, N_{ht}^{-1} N_{ht}' - m_t' U_\ell(\cdot) + \mu_{2t}\, \alpha_s\, Y_{st}\, N_{\ell t}^{-1} N_{\ell t}' \\
&-\mu_{1,t} m_t' \left(\xi_h^{-\frac{1}{\nu_h}} C_{ht}^{\frac{1}{\nu_h}} + \eta_h^{-1} C_{hht} - \xi_\ell^{-\frac{1}{\nu_\ell}} C_{\ell t}^{\frac{1}{\nu_\ell}} - \eta_\ell^{-1} C_{\ell ht} \right) \\
&-\mu_{2,t} m_t^{'} \left(C_{hmt} - C_{\ell mt} \right) \\
0 =& U_h(\cdot) + \mu_{1t}\, \alpha_g\, Y_{gt}\, N_{ht}^{-1} N_{ht}' (m_t')^{-1} - U_\ell(\cdot) + \mu_{2t}\, \alpha_s\, Y_{st}\, N_{\ell t}^{-1} N_{\ell t}' (m_t')^{-1} \\
&-\mu_{1,t} \left(\xi_h^{-\frac{1}{\nu_h}} C_{ht}^{\frac{1}{\nu_h}} + \eta_h^{-1} C_{hht} - \xi_\ell^{-\frac{1}{\nu_\ell}} C_{\ell t}^{\frac{1}{\nu_\ell}} - \eta_\ell^{-1} C_{\ell ht} \right) \\
&-\mu_{2,t} \left(C_{hmt} - C_{\ell mt} \right)
\end{aligned}
$$

Recall that $m_t = e^{-\lambda s_t}$ to compute $m_t^{'} = -\lambda e^{-\lambda s_t}$ and other relevant expressions

$$
\begin{array}{llll}
N_{\ell t} = \chi_\ell (1 - e^{-\lambda s_t}) & N_{\ell t}^{'} = \chi_\ell \lambda e^{-\lambda s_t} & N_{\ell t}^{'} (m_t^{'})^{-1} = -\chi_\ell & N_{\ell t}^{-1} N_{\ell t}^{'} (m_t^{'})^{-1} = \dfrac{-1}{1 - e^{-\lambda s_t}} \\
N_{ht} = \dfrac{\chi_h}{\lambda} (1 + \lambda s_t) e^{-\lambda s_t} & N_{ht}^{'} = -\chi_h \lambda s_t e^{-\lambda s_t} & N_{ht}^{'} (m_t^{'})^{-1} = \chi_h s_t & N_{ht}^{-1} N_{ht}^{'} (m_t^{'})^{-1} = \dfrac{\lambda s_t e^{\lambda s_t}}{1 + \lambda s_t}
\end{array}
$$

$$
\begin{aligned}
0 =& U_h(\cdot) + \mu_{1t}\, \alpha_g\, Y_{gt} \left[\frac{\lambda s_t e^{\lambda s_t}}{1 + \lambda s_t} \right] - U_\ell(\cdot) - \mu_{2t}\, \alpha_s\, Y_{st} \left[\frac{1}{1 - e^{-\lambda s_t}} \right] \\
&-\mu_{1,t} \left(\xi_h^{-\frac{1}{\nu_h}} C_{ht}^{\frac{1}{\nu_h}} + \eta_h^{-1} C_{hht} - \xi_\ell^{-\frac{1}{\nu_\ell}} C_{\ell t}^{\frac{1}{\nu_\ell}} - \eta_\ell^{-1} C_{\ell ht} \right) \\
&-\mu_{2,t} \left(C_{hmt} - C_{\ell mt} \right)
\end{aligned}
$$

Combining the last two equations we obtain

$$\begin{aligned}
\mu_{3t}\frac{1}{q_t} &= \beta\,\mathbb{E}_t\left[\mu_{3,t+1}\left(\frac{1-\delta(u_{t+1})}{q_{t+1}}\right)+\mu_{4,t+1}u_{t+1}\right]\\
\frac{\mu_{4t}}{\delta'(u_t)} &= \beta\,\mathbb{E}_t\left[\frac{\mu_{4,t+1}q_{t+1}}{\delta'(u_{t+1})}\left(\frac{1-\delta(u_{t+1})}{q_{t+1}}\right)+\mu_{4,t+1}u_{t+1}\right]\\
\frac{\mu_{4t}}{u_t^{\tau-1}} &= \beta\,\mathbb{E}_t\,\frac{\mu_{4,t+1}}{u_{t+1}^{\tau-1}}\left[1+(1-\frac{1}{\tau})u_{t+1}^{\tau}\right]\\
\mu_{1t}\frac{Y_{gt}K_{gt}^{-1}}{u_t^{\tau-1}} &= \beta\,\mathbb{E}_t\,\mu_{1,t+1}\frac{Y_{g,t+1}K_{g,t+1}^{-1}}{u_{t+1}^{\tau-1}}\left[1+(1-\frac{1}{\tau})u_{t+1}^{\tau}\right]
\end{aligned}$$

Rearranging the rest of the FOCs, we obtain the following

$$\frac{\mu_{1t}}{\mu_{2t}} = \left[\frac{\omega_g M_{st}}{\omega_s M_{gt}}\right]^{\frac{1}{\varepsilon}} \tag{6.0.2}$$

$$\mu_{1t}M_{gt}+\mu_{2t}M_{st} = \mu_{3t}I_t \tag{6.0.3}$$

$$\frac{\mu_{1t}}{\mu_{2t}} = \frac{(1-\alpha_s)Y_{st}K_{gt}}{(1-\alpha_g)Y_{gt}K_{st}} \tag{6.0.4}$$

$$\mu_{1t}(1-\alpha_g)Y_{gt}+\mu_{2t}(1-\alpha_s)Y_{st} = \mu_{4t}u_tK_t \tag{6.0.5}$$

$$\mu_{3t}\frac{\delta'(u_t)}{q_t} = \mu_{4t} \tag{6.0.6}$$

$$\mu_{1t}\frac{Y_{gt}K_{gt}^{-1}}{u_t^{\tau-1}} = \beta\mathbb{E}_t\mu_{1,t+1}\frac{Y_{g,t+1}K_{g,t+1}^{-1}}{u_{t+1}^{\tau-1}}\left[1+(1-\frac{1}{\tau})u_{t+1}^{\tau}\right] \tag{6.0.7}$$

Finally, work on the second last equation in the system:

$$\begin{aligned}
\frac{\mu_{3t}}{\mu_{2t}}\frac{\delta'(u_t)}{q_t} &= \frac{\mu_{4t}}{\mu_{2t}}\\
\frac{\delta'(u_t)}{q_t}\frac{1}{I_t}\left(\frac{\mu_{1t}}{\mu_{2t}}M_{gt}+M_{st}\right) &= \frac{1}{u_tK_t}\left(\frac{\mu_{1t}}{\mu_{2t}}(1-\alpha_g)Y_{gt}+(1-\alpha_s)Y_{st}\right)\\
\frac{\delta'(u_t)}{q_t}\frac{1}{I_t}\left(\left[\frac{\omega_g M_{st}}{\omega_s M_{gt}}\right]^{\frac{1}{\varepsilon}}M_{gt}+M_{st}\right) &= \frac{1}{u_tK_t}\left(\frac{(1-\alpha_s)Y_{st}K_{gt}}{(1-\alpha_g)Y_{gt}K_{st}}(1-\alpha_g)Y_{gt}+(1-\alpha_s)Y_{st}\right)\\
\frac{\delta'(u_t)}{q_t}\frac{1}{I_t}\left[\frac{M_{st}}{\omega_s}\right]^{\frac{1}{\varepsilon}}\left[\frac{I_t}{A_{It}}\right]^{\frac{\varepsilon-1}{\varepsilon}} &= \frac{1}{u_tK_t}\left(\frac{(1-\alpha_s)Y_{st}K_{gt}}{K_{st}}+(1-\alpha_s)Y_{st}\right)\\
\frac{u_t^{\tau-1}}{q_t}\left[\frac{M_{st}}{\omega_s I_t}\right]^{\frac{1}{\varepsilon}}A_{It}^{\frac{1-\varepsilon}{\varepsilon}} &= \frac{(1-\alpha_s)Y_{st}}{K_{st}}
\end{aligned}$$

The complete system (together with definitions) reads:

$$-\Phi_{1h}C_{ht}^{-\gamma} + \frac{\xi_h^{-\frac{1}{\nu_h}}}{\nu_h}\mu_{1t}C_{ht}^{\frac{1}{\nu_h}-1} = \eta_h^{-1}\left(\frac{\epsilon_{hh}-\sigma_h}{\sigma_h-1}\right)\left(\frac{C_{hht}}{C_{ht}}\right)\mu_{1t} + \left(\frac{\epsilon_{hm}-\sigma_h}{\sigma_h-1}\right)\left(\frac{C_{hmt}}{C_{ht}}\right)\mu_{2t} \tag{6.0.8}$$

$$\frac{\eta_h^{-1}\mu_{1t}}{\mu_{2ht}} = \left(\frac{\Omega_{hh}}{\Omega_{hm}}\right)^{\frac{1}{\sigma_h}} C_{ht}^{\frac{\epsilon_{hh}-\epsilon_{hm}}{\sigma_h}} \left(\frac{C_{hht}}{C_{hmt}}\right)^{-\frac{1}{\sigma_h}} \tag{6.0.9}$$

$$-\Phi_{1\ell}C_{\ell t}^{-\gamma} + \frac{\xi_\ell^{-\frac{1}{\nu_\ell}}}{\nu_\ell}\mu_{1t}C_{\ell t}^{\frac{1}{\nu_\ell}-1} = \eta_\ell^{-1}\left(\frac{\epsilon_{\ell h}-\sigma_\ell}{\sigma_\ell-1}\right)\left(\frac{C_{\ell ht}}{C_{\ell t}}\right)\mu_{1t} + \left(\frac{\epsilon_{\ell m}-\sigma_\ell}{\sigma_\ell-1}\right)\left(\frac{C_{\ell mt}}{C_{\ell t}}\right)\mu_{2t} \tag{6.0.10}$$

$$\frac{\eta_\ell^{-1}\mu_{1t}}{\mu_{2\ell t}} = \left(\frac{\Omega_{\ell h}}{\Omega_{\ell m}}\right)^{\frac{1}{\sigma_\ell}} C_{\ell t}^{\frac{\epsilon_{\ell h}-\epsilon_{\ell m}}{\sigma_\ell}} \left(\frac{C_{\ell ht}}{C_{\ell mt}}\right)^{-\frac{1}{\sigma_\ell}} \tag{6.0.11}$$

$$\frac{\mu_{1t}}{\mu_{2t}} = \left[\frac{\omega_g M_{st}}{\omega_s M_{gt}}\right]^{\frac{1}{\varepsilon}} \tag{6.0.12}$$

$$\frac{\mu_{1t}}{\mu_{2t}} = \frac{(1-\alpha_s)Y_s K_g}{(1-\alpha_g)Y_g K_s} \tag{6.0.13}$$

$$\frac{u_t^{\tau-1}}{q_t}\left[\frac{M_{st}}{\omega_s I_t}\right]^{\frac{1}{\varepsilon}} A_{It}^{\frac{1-\varepsilon}{\varepsilon}} = \frac{(1-\alpha_s)Y_{st}}{K_{st}} \tag{6.0.14}$$

$$\mu_{1t}\left(\frac{Y_{gt}K_{gt}^{-1}}{u_t^{\tau-1}}\right) = \beta\,\mathbb{E}_t\,\mu_{1,t+1}\left(\frac{Y_{g,t+1}K_{g,t+1}^{-1}}{u_{t+1}^{\tau-1}}\right)\left[1+(1-\frac{1}{\tau})u_{t+1}^{\tau}\right] \tag{6.0.15}$$

$$Y_{gt} = M_{gt} + m_t\left(\xi_h^{-\frac{1}{\nu_h}}C_{ht}^{\frac{1}{\nu_h}} + \eta_h^{-1}C_{hht}\right) + (1-m_t)\left(\xi_\ell^{-\frac{1}{\nu_\ell}}C_{\ell t}^{\frac{1}{\nu_\ell}} + \eta_\ell^{-1}C_{\ell ht}\right) \tag{6.0.16}$$

$$Y_{st} = M_{st} + m_t C_{hmt} + (1-m_t)C_{\ell mt} \tag{6.0.17}$$

$$I_t = \frac{K_{t+1}}{q_t} - [1-\delta(u_t)]\frac{K_t}{q_t} \tag{6.0.18}$$

$$u_t K_t = K_{gt} + K_{st} \tag{6.0.19}$$

$$Y_{gt} = A_{gt}K_{gt}^{1-\alpha_g}N_{ht}^{\alpha_g} \tag{6.0.20}$$

$$Y_{st} = A_{st}K_{st}^{1-\alpha_s}N_{\ell t}^{\alpha_s} \tag{6.0.21}$$

$$I_t = A_{It}\left(\omega_g^{\frac{1}{\epsilon}}M_{gt}^{\frac{\epsilon-1}{\epsilon}} + \omega_s^{\frac{1}{\epsilon}}M_{st}^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}} \tag{6.0.22}$$

$$1 = \sum_{i\in\{g,s\}}\Omega_{hi}^{\frac{1}{\sigma_h}}C_{ht}^{\frac{\epsilon_{hi}-\sigma_h}{\sigma_h}}C_{hit}^{\frac{\sigma_h-1}{\sigma_h}} \tag{6.0.23}$$

$$1 = \sum_{i\in\{g,s\}}\Omega_{\ell i}^{\frac{1}{\sigma_\ell}}C_{\ell t}^{\frac{\epsilon_{\ell i}-\sigma_\ell}{\sigma_\ell}}C_{\ell it}^{\frac{\sigma_\ell-1}{\sigma_\ell}} \tag{6.0.24}$$

$$N_{ht} = \frac{\chi_h}{\lambda}(1+\lambda s_t)e^{-\lambda s_t} \tag{6.0.25}$$

$$N_{\ell t} = \chi_\ell(1-e^{-\lambda s_t}) \tag{6.0.26}$$

$$m_t = e^{-\lambda s_t} \tag{6.0.27}$$

and

$$
\begin{aligned}
0 =& U_h(\cdot) + \mu_{1t}\,\alpha_g\,Y_{gt}\left[\frac{\lambda s_t e^{\lambda s_t}}{1+\lambda s_t}\right] - U_\ell(\cdot) - \mu_{2t}\,\alpha_s\,Y_{st}\left[\frac{1}{1-e^{-\lambda s_t}}\right] \\
&-\mu_{1,t}\left(\xi_h^{-\frac{1}{\nu_h}} C_{ht}^{\frac{1}{\nu_h}} + \eta_h^{-1} C_{hht} - \xi_\ell^{-\frac{1}{\nu_\ell}} C_{\ell t}^{\frac{1}{\nu_\ell}} - \eta_\ell^{-1} C_{\ell ht}\right) \\
&-\mu_{2,t}\left(C_{hmt} - C_{\ell mt}\right)
\end{aligned}
$$

There are 19 equations and 19 variables:

$$
C_h, C_{hh}, C_{hm}, C_\ell, C_{\ell h}, C_{\ell m}, N_h, N_\ell, u, M_g, M_s, K_g, K_s, K, Y_g, Y_s, I, \mu_1, \mu_2
$$