The Role of Correlated Investment-Skill Shocks in Business Cycles*

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Abstract

In this paper, I study the effects of an exogenous investment shock that is correlated with the skill distribution in the population. In the model, agents, who have heterogeneous productivities, are sorted into different occupations based on their comparative advantages over the business cycle. I also assume that each occupation is associated with a different consumption bundle. This latter assumption implies that as agents switch between low and high-skill jobs their consumption behavior changes in a discrete fashion as well, which in turn gives rise to non-linear income expansion paths at the aggregate level without assuming non-homothetic preferences at the individual level. In this environment, I show that after a positive investment shock, the fraction of agents working in the goods sector increases on impact and those who start working in the goods sector replace their service intensive consumption baskets with goods intensive ones, and therefore the demand for goods increases more relative to the services along the expansion, which in return results in even more agents being hired in the goods sector. I show quantitatively that this endogenous mechanism alone can generate sizable amplification and persistence, even without labor-leisure choice, in contrast to the standard models. The model also suggests an alternative resolution to what is known as the productivity puzzle in the literature. This is because the one-to-one link between output and productivity does not exist in the model, in contrast to the standard models. Instead, the productivity is determined by the aggregate skill distribution and how agents are sorted between two sectors over the business cycle.

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1 Introduction

In this paper, I consider an economy in which production of each good requires a different task to be performed and agents have heterogeneous skills over the tasks. In this environment agents specialize in one of the tasks based on their comparative advantages[1] and, just like in Karadas (2017), this specialization gives rise to the set of goods that an agent contributes to its production and her consumption bundle being different. This discrepancy induces the type of interdependencies across the population as mentioned there. However, unlike the model in Karadas (2017), agents can switch between the sectors over the business cycle, so the complete specialization assumption is relaxed here. In particular, I assume that there are different consumption bundles (or different lifestyles as referred to in Karadas (2017)) and agents choose one based on the type of skill they supply. Since agents might end up supplying different skills over the business cycle, under the stated assumption, they change their consumption bundles as well, which in return gives rise to an endogenous variation in the demand composition over time. In this paper, I study the contribution of this endogenous variation in the demand composition to the propagation of economic shocks, in addition to the mechanisms that were already at work in Karadas (2017).

This additional mechanism that I incorporate into the model can be further motivated by the following example. If the production of a certain good, say it is a capital intensive good, requires more than one task to be performed, and agents are heterogeneous in their productivity in each of these tasks, an efficiency seeking firm would seek to achieve an optimal assignment of tasks to skills. In particular, a firm would not want to hire a high-skill agent for a simple task that could be performed equally well by a low-skill agent, and this is where specialization appears at the firm level. But what really matters in terms of the business cycle is the following: in response to a negative shock, firms might find it optimal to fire low-skill agents first, as high-skill agents can substitute for them perfectly in most cases, as long as high-skill agents are willing to perform a wider task range, maybe even for a slightly lower wage.[2] But if the low-skill agents' propensity to consume capital intensive goods is higher than high skill agents, then losing one customer among the low-

[1] See Acemoglu and Autor (2011) for a recent survey paper on task based models.
[2] In Section 4, I give an example of a simple production structure with these features.

skill agents is more consequential in terms of the total demand for that good than losing one from high-skill agents. Therefore, the interaction between consumption and production sides of the economy in terms of their factor contents is at the heart of the discussion in this paper.

In the following sections, after a discussion on the way I conceptualize the consumption side of the economy, I present a very simple model featuring some of these ingredients and I conclude the paper with a quantitative assessment of the model.

A Discussion of Consumption Categories

As discussed in Karadas (2017), instead of making the labor-leisure choice a central point in the modeling, I model agents as if they choose an optimal consumption bundle under normal circumstances as a function of their relative positions in the economy (their wage income in my model). Of course, the set of possible consumption patterns can be specified in many different ways but, I think in terms of its relevance to the business cycle research, the following two group categorization is particularly useful in investigating interdependencies generated by the division of labor across the population.

- **Category of Goods (C_g):** include all the goods that do not require someone else's labor service at the moment of consumption. However, they might be of various qualities. For simplicity of the exposition in this section, assume they come only in two versions; goods are either basic goods (C_b) or customized goods (C_c). The main difference between these two versions of goods is their skill intensity, the latter has a greater high-skill content.

- **Category of Services (C_s):** On the other hand, goods consumed under C_s should satisfy the following condition: they can potentially be met by the consumer themselves once the necessary inputs are purchased at the market, and if they are not met by the consumer, they should require someone else's labor service at the time of consumption. In the former case, the consumer produces the service at home but has to decide between the basic or customized inputs, C_b and C_c, to produce the service goods at home (C_h). On the other hand, in the latter case the consumer purchases the service from the market (C_m), which is produced by competitive firms using again

C_b and C_c and labor, in addition. For instance, consider someone who would like to learn to play the guitar. If a computer software is purchased for that, the service is C_h, and if a music teacher is hired the service consumed is C_m.

For simplicity, assume that only basic goods and low-skill labor are required to produce market services C_m, which approximately corresponds to service occupations. For example, babysitting, dining out or recreational services can be considered as examples of C_m. However, in principle C_m might include some other input combinations as well. For example, any kind of non-compulsory educational service should be considered as a market service in this categorization, and be counted under C_m.

In summary, consumers derive utility from goods (C_g) and services (C_s) according to a consumption aggregate

$$C_t = C(C_{gt}, C_{st}),$$

and in return both C_{gt} and C_{st} have two versions:

$$C_{gt} \in \{C_{bt}, C_{ct}\}, \quad C_{st} \in \{C_{ht}, C_{mt}\},$$

and agents can consume either the basic version of the goods (C_{bt}) or the customized version (C_{ct}). Similarly, agents choose between market produced service goods, C_{mt}, and home produced service goods, C_{ht}.

In the rest of this paper, for simplicity I assume that only one good from each category is consumed and I will label them as goods and services, which are denoted by C_g and C_s, respectively. I Furthermore, I assume that the consumption aggregator is Cobb-Douglas:

$$C_{it}^{k} \equiv C_{igt}^{\gamma_k} C_{ist}^{1-\gamma_k}, \qquad k \in \{L, H\}, \tag{1}$$

where subscript i denotes the identity of an agent and k her type.

Note that I assume preferences of an agent depends her type, i.e. whether she is employed as a high skill or low skill worker. In the rest of this section, I present some empirical evidence for this assumption and provide an extended discussion on its implications in the model.

A Digression: The Cross-sectional Consumption Pattern

The cross-sectional data clearly shows that the expenditure shares of goods and services vary across the population at a point in time: as the income level increases, consumption bundles contain more varieties and a higher fraction of services purchased at the market (see Chai, Rohde and Silber (2015)).[3]

However, this cross-sectional variability of consumption bundles should not to be confused with *preference heterogeneity* because it is a *preference pattern* that arises mostly due to the income and wealth inequality. In order to develop some motivation for the assumption I make in this paper, let us use the same consumption aggregator above $C_t \equiv C_{gt}^{\gamma} C_{st}^{1-\gamma}$. Also in instead of assuming fixed expenditure shares of goods and services as implied by Cobb-Douglas preferences, suppose that there is a cross-sectional preference pattern such that agents approximately move along the same pattern as their income rises. For example, suppose that the share of expenditure on goods is given by the following time-invariant function

$$\gamma = \gamma(y)\,,$$

where y is income of the agent. In other words, an agent with income y spends $\gamma(y)$ fraction of her total consumption expenditures on goods, and $1-\gamma(y)$ on services. This assumption also entails that agents just adopt the consumption bundles of higher income agents as their income increases.[4] As an example, consider the following functional form

$$\gamma(y) = ae^{-by}, \qquad a \in (0,1), \quad y \in [0,\infty), \tag{2}$$

where the lowest income level is normalized to 0. According to this invariant population preference distribution, the expenditure share of goods is a for the lowest income agent, and $ae^{-b\overline{y}}$ for the highest income agent as shown in the figure below. If the level of income of the highest type rises, say from $\overline{y}$ to $\overline{y}'$, then the expenditure share of goods decreases

[3] Chai, Rohde and Silber (2015) list three stylized facts in their survey paper: (i) At low income levels, spending diversity is low as food expenditure dominates spending, (ii) As household income grows, spending diversity increases via reductions in the budget share of food spending and increases in non-food expenditure. (iii) Individual household spending becomes more diversified as income rises.

[4] Alternatively, the population preference distribution can be modeled as a function of the agent's rank in terms of their income.

from $ae^{-b\overline{y}}$ to $ae^{-b\overline{y}'}$. Similarly, other agents move towards the right depending on their new income levels.

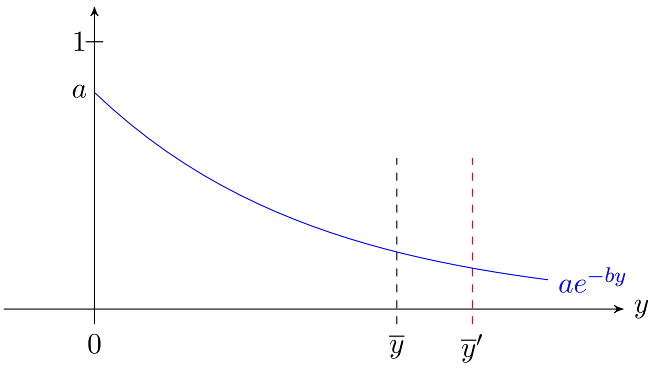


Figure 1: Cross-sectional Consumption Pattern

In this framework, once a preference distribution is specified then the average factor content of demand can be easily calculated. For example, for a the preference distribution $\gamma(y)$ the average share of expenditure on goods can be calculated as follows

$$\overline{\gamma}(\overline{y}) = \int_0^{\overline{y}} \gamma(y) f(y) dy,$$

where $f(y)$ denotes the income distribution. This expression, in return, can be used to approximate how the factor content of aggregate demand varies as the society becomes more affluent. For instance, under a reasonable parametrization of the income distribution and the exponential specification above for $\gamma(y)$, it can be shown that the average share of expenditure on goods out of total expenditures decreases as the income level increases, i.e. $\partial\gamma/\partial\overline{y} < 0$.

Assuming a time invariant cross-sectional consumption pattern $\gamma(y)$ is akin to having non-homothetic preferences in the model as I did in Karadas (2017) by using the following Stone-Geary preferences,

$$C_t \equiv \left(C_{gt} - \overline{C}_g\right)^{\gamma} \left(C_{st} - \overline{C}_s\right)^{1-\gamma}.$$

If agents are heterogenous in terms of their income, Stone-Geary preferences also induce income dependent expenditure shares across the population at a certain point in time. However, it is hard to obtain a target or empirically observed pattern of cross sectional expenditure distribution under these preferences as the realized expenditure shares depend on prices which are in turn endogenous to the model.

In terms of general equilibrium thinking, this assumption should cause no trouble since in a business cycle model, the main concern should be to replicate the main aggregate statistics and eventually provide some predictions for the course of aggregate variables, not to track the evolution of the exact consumption bundles of agents over the business cycle. I claim that imposing such an empirically robust discipline in multi-sector business cycle models can only increase their applicability. Methodologically this paper can be considered as an attempt to incorporate a significant amount of heterogeneity into the model without rendering it intractable. Using a time-invariant preference structure simplifies the analysis in the following sense: since the agents simply move along a time-invariant curve, we only need to keep track of the measure of agents who push the current frontier of the consumption bundle further and the measure of agents who moved forward from the bottom part of the preference distribution. The difference between the measures of these two groups gives a good idea about the total change in the factor content of goods and services consumed as the economy transits from one point to another.

Finally, to describe the preferences of an agent with income y a slightly more general form can also be assumed:

$$C_t \equiv [\gamma(C_{gt})^\epsilon + (1-\gamma)(C_{st})^\epsilon]^{\frac{1}{\epsilon}} \ ,$$

where $\gamma = \gamma(y)$ is income dependent consumption share that agents take as a parameter. In this formulation, ϵ can be interpreted as the measurement error incurred in calculating the expenditure shares for agents with income y. Under this interpretation Cobb-Douglas specification assumes no measurement error.

2 Model

In the model, there is a continuum of measure one agents $s \in [0, 1]$ who have homogenous skills in producing services but have heterogenous skill endowments in the goods production, and agents are sorted into one of the sectors according to their comparative advantage. On the production side, there are two representative firms producing basic goods[5] and services, respectively. In the next three sections, after describing the problem of households and firms, I define an equilibrium for the economy.

Households' Problem

Each household $s \in [0, 1]$ is endowed with a vector of efficiency units as a low and high skill worker,[6] denoted by $e_\ell(s)$ and $e_h(s)$, respectively. Workers have the same efficiency units (normalized to unity) at performing low skill jobs, i.e. $e_\ell(s) = 1, \forall s$. However, agents have heterogeneous skills in performing high skill jobs. Since this is the only heterogeneity among the agents, I will use the same letter to denote the agent's efficiency units in goods production, i.e. s equal the worker's skill in high skill jobs, measured in efficiency units. I assume that skills s are distributed according to a density function $f(s)$ across the population.

Each period agents self-select into which skill they would like to supply given wages for low and high skill jobs. Each worker supplies one unit of labor inelastically to the task offering the highest income level given her endowment, s. An agent i who chooses to work in job j, $j \in \{L, H\}$, receives w_{jt} for each efficiency unit. As usual in models with efficiency units labor income increases in proportion to efficiency units supplied at the equilibrium wage, which, in this model, will be $w_{jt}e_{jt}(i)$, where $e_{jt}(i)$ is agent i's efficiency units in job (task) j, and w_{jt} is the equilibrium wage for one efficiency unit in job j at time t.

Aggregate capital stock, K_t, is owned by the households. Therefore, agent i also makes a saving plan by renting capital, k_{it}, to the firm at the rental price R_t, and capital depreciates at the rate δ. The initial endowment of the individual i is denoted by k_0^i. I assume that agents are symmetric in terms of ownership rights.

[5] I use the terms *basic goods* and *goods* interchangably in the paper unless there is a risk of confusion.

[6] In task-based approach these skills alternatively labeled *manual*, *rotuine*, *abstract*, etc. See Acemoglu and Autor (2011).

I assume that markets are complete. In particular, an agent can insure herself against aggregate uncertainty by trading state contingent claims, $\{b_{it}(\theta)\}_{\theta\in\Theta}$, where Θ denotes the set of all possible states. The claim of type $\theta \in \Theta$ costs $q_t(\theta)$ in period t and entitles the buyer one unit of basic goods C_g in period $t+1$ if the state $\theta \in \Theta$ is realized and zero otherwise.

Consumer i derives utility from goods (C_{igt}) and services (C_{ist}) according to Cobb-Douglas consumption aggregate:

$$C_{it}^k \equiv C_{igt}^{\gamma_k} C_{ist}^{1-\gamma_k}\,, \qquad k \in \{L, H\}\,.$$

Note that I assume preferences of an agent depends whether he is employed as a high or low skill labor.

Finally, I assume that agents supply labor inelastically and have the following utility function

$$U(C_{igt}, C_{ist}) = \frac{1}{1-\sigma} C_{it}^{1-\sigma},$$

where $\sigma > 0$.

Household $i \in [0,1]$ seeks to maximize the expected sum of momentary utilities $U(C_{it})$, discounted at the rate β by choosing a path for consumption vector $\{C_{it}\}_{t=0}^{\infty}$, capital $\{k_{it}\}_{t=1}^{\infty}$, capacity utilization $\{u_t\}_{t=1}^{\infty}$ and securities $\{b_{it}\}_{t=1}^{\infty}$:

$$\max_{\{C_{it}, I_{it}, u_t, \{b_{it}(\theta)\}_{\theta\in\Theta}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{C_{it}^{1-\sigma}}{1-\sigma}\,, \quad (\sigma > 0) \tag{3}$$

subject to

$$(i)\, C_{igt} + P_{st} C_{ist} + I_{i,t} + \int_{\Theta} q_t(\theta) b_{i,t+1}(\theta) d\theta = (u_t k_{it}) R_t + w_{jt} e_{jt}(i) + b_{it}(\theta_t), \tag{4}$$

$$(ii)\, k_{i,t+1} = (1 - \delta(u_t)) k_{i,t} + I_{i,t} q_t^I \tag{5}$$

$$(iii)\, w_{jt} e_{jt}(i) = \max\{w_{\ell t}, w_{ht} e_{ht}(i)\} \tag{6}$$

$$(iv)\, \text{given } k_{i0} \text{ and } b_{i0}. \tag{7}$$

where I use the normalization $P_{gt} = 1$ and $C_{it} = (C_{igt}, C_{ist})$ is time t consumption vector of goods and services, respectively. The condition $w_{jt}e_{jt}(i) = \max\{w_{\ell t}, w_{ht}e_{ht}(i)\}$ ensures that consumers self-select into the occupations optimally given the prevailing wages and their efficiency units in each job.

Note that the model assumes variable capacity utilization and there is an aggregate uncertainty about the technological shift variable q_t^I in the model. I again follow the framework in Greenwood, Hercowitz and Huffman (1988) in modeling the investment as I did in Karadas (2017), so I do not repeat that part here.

Firms' Problem

On the production side, there are two representative firms producing basic goods (Y_{gt}) and service goods (Y_{st}). Each firm solves the usual static problems that I outline briefly in this section.

Production of basic goods requires capital and high skill labor and therefore the representative firm that produces (basic) goods, Y_{gt}, solves the following problem:

$$\max_{\tilde{K}_t, H_t} \; A_{gt}\tilde{K}_t^{1-\alpha_g} H_t^{\alpha_g} - R_t\tilde{K}_t - w_{ht}H_t \, , \tag{8}$$

where $\tilde{K}$ is the capital services rented by the firm and H_t is the total high skill labor measured in efficiency units.

First order conditions of the firm's problem:

$$R_t(u_tK_t) = (1-\alpha_g)Y_{gt} \tag{9}$$

$$w_{ht}N_{ht} = \alpha_g Y_{gt} \, , \tag{10}$$

where the equilibrium condition for capital $\tilde{K}_t = u_tK_t$ is already incorporated.

On the other hand, the production of services requires basic goods and low skill labor to produce output and consequently the representative firm that produces service goods, Y_{st}, solves:

$$\max_{X_{st}, L_t} \; P_{st}\, A_{st}X_t^{1-\alpha_s} L_t^{\alpha_s} - X_t - w_{\ell t}L_t \tag{11}$$

and the first order conditions of this problem determines the demand for (basic) goods X_t and the wage rate for low-skill labor:

$$X_t = (1-\alpha_s)P_{st}Y_{st} \tag{12}$$

$$w_{\ell t}L_t = \alpha_s P_{st}Y_{st} \tag{13}$$

Equilibrium

An equilibrium for this economy can be defined as follows: agents and firms solve their respective problems as stated in the previous section and market clearing conditions hold in each market:

- The market clearing conditions for Goods and Service sectors

  $$\begin{aligned} Y_{gt} &= C_{gt} + I_t + X_t\,, \\ Y_{st} &= C_{st}\,, \end{aligned}$$

  where

  $$\begin{aligned} C_{gt} &= m_t C_{hgt} + (1-m_t)C_{\ell gt} \\ C_{st} &= m_t C_{hst} + (1-m_t)C_{\ell st}\,, \end{aligned}$$

  m_t is the measure of agents working in high skill jobs in period t.

- Labor market clearing conditions:

  $$\begin{aligned} L_t &= \int_0^{s_t} e_\ell(s)f(s)\,ds \\ H_t &= \int_{s_t}^{\infty} e_h(s)f(s)\,ds\,, \end{aligned}$$

- The law of motion for capital:

  $$\begin{aligned} K_{t+1} &= (1-\delta(u_t))\,K_t + I_t q_t^I \\ K_t &= \int_{i\in[0,1]} k_{i,t}\,di\,, \end{aligned}$$

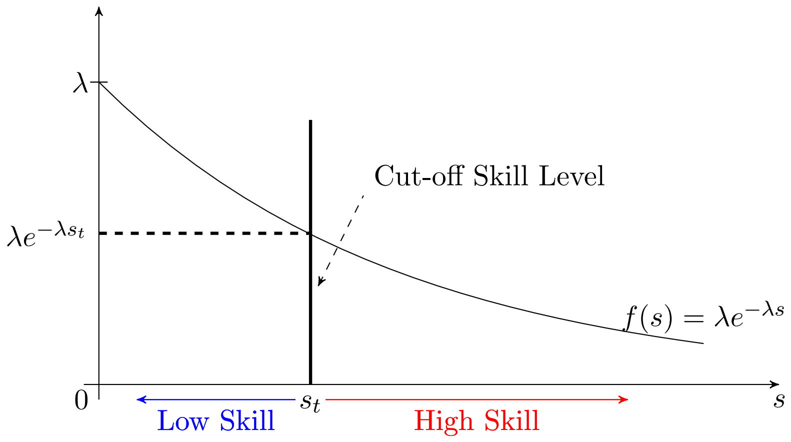


Figure 2: Skill Distribution

Definition of GDP and Productivity

I define the GDP as the total gross value of consumption and investment expressed in terms of (basic) goods:

$$Y_t = C_t + I_t \,,$$

where $C_t = C_{gt} + P_t C_{st}$ is the total consumption.

I define the labor productivity of sector, say A, in terms of efficiency units employed in the production:

$$\textbf{Labor Productivity} = \frac{\text{Total Output of Sector A}}{\text{Total Labor Employed in Sector A (in Efficiency Units)}}$$

Planner's Problem

For simplicity, I assume that efficiency units (skills) are distributed exponentially on the interval $[0, \infty]$ with the density function $f(s) = \lambda e^{-\lambda s}$. This assumption together with the condition $w_{jt} e_{jt}(i) = \max\{w_{\ell t}, w_{ht} e_{ht}(i)\}$ in the consumer's problem guarantees that there is a cut-off skill level s_t such that agents with skill below s_t take low skill jobs and the rest work as high skill workers. With this particular functional form the measure (actually proportion [7]) of high skill agents is $m_t = e^{-\lambda s_t}$ (see Figure 2).

[7] Think of the CDF of f, $F(s) = 1 - e^{-\lambda s}$. Workers with $s \geq s_t$ take high skill jobs, which is of measure $e^{-\lambda s_t}$.

Note that for a given cut-off skill level, the allocation of total labor between basic goods and service sector is uniquely determined and consequently the existence of a unique equilibrium is guaranteed for each period under the standard assumptions. This observation also suggests that instead of studying this economy, I can focus on a simpler problem in which a social planner determines the cut-off skill level optimally, given the preference distribution in the population and the stochastic structure of the economy.

The planner's problem is to find a pair of welfare weights $\{\lambda_\ell, \lambda_h\}$ and a sequence of contingency plans $\{C_{\ell gt}, C_{\ell st}\}$ for L-type workers and $\{C_{hgt}, C_{hst}\}$ for H-type workers and for aggregate allocations $\{K_t, X_{st}, u_t\}$ such that the sum of individual utilities that are weighted by the corresponding welfare weights is maximized:

$$
\begin{aligned}
\max_{\left\{\{C_{igt},C_{ist}\}_{i\in\{\ell,h\}},X_{st},K_{t+1},u_t,s_t\right\}_{t=0}^{\infty}} \quad & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big\{ \lambda_\ell m_t\, U_\ell(\cdot) + \lambda_h(1-m_t) U_h(\cdot) \Big\} \\
& + \mu_{1,t}\left[Y_{gt} - C_{gt} - X_{st} - I_t\right] \\
& + \mu_{2,t}\left[Y_{st} - C_{st}\right] \\
& + \mu_{3,t}\left[C_{gt} - m_t C_{hgt} - (1-m_t) C_{\ell gt}\right] \\
& + \mu_{4,t}\left[C_{st} - m_t C_{hst} - (1-m_t) C_{\ell st}\right] \\
& + \mu_{5,t}\left[K_{t+1} - (1-\delta(u_t))K_t - I_t\right]
\end{aligned}
$$

where

$$
\begin{aligned}
Y_{gt} &= A_{gt}\big[u_t K_t\big]^{1-\alpha_g}\big[H_t\big]^{\alpha_g} \\
Y_{st} &= A_{st} X_{st}^{1-\alpha_s} L_t^{\alpha_s} \\
L_t &= \int_0^{s_t} f(s)\, ds \\
H_t &= \int_{s_t}^{\infty} e_h(s) f(s)\, ds\,,
\end{aligned}
$$

and $m_t = 1 - F(s_t)$ is the measure of high skill agents and $\delta(u_t) = (1/\tau)u_t^\tau$ is an increasing, convex function with $\tau > 1$.

Note that once the threshold skill level s_t is chosen, labor is uniquely allocated between the two sectors and also the resulting measure m_t of high skill agents determine the demand for each good.

In Appendix, I obtain expressions for L_t and H_t in terms of the parameter of the skill distribution λ and the cut off skill level s_t. Using these expressions and after a few substitutions the problem becomes:

$$\begin{aligned}
\max_{\left\{\{C_{igt},C_{ist}\}_{i\in\{\ell,h\}},X_{st},K_{t+1},u_t,s_t\right\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big\{ & \lambda_h m_t U_h(C_{hgt},C_{hst}) + \lambda_\ell(1-m_t)U_\ell(C_{\ell gt},C_{\ell st}) \\
& +\mu_{1,t}\Big[Y_{gt} - m_t C_{hgt} - (1-m_t)C_{\ell gt} \\
& \qquad - X_{st} - K_{t+1}/q_t + [1-\delta(u_t)]K_t/q_t \Big] \\
& +\mu_{2,t}\left[Y_{st} - m_t C_{hst} - (1-m_t)C_{\ell st}\right] \Big\},
\end{aligned}$$

where

$$\begin{aligned}
Y_{gt} &= A_{gt}\big[u_t K_t\big]^{1-\alpha_g}\left[\frac{1}{\lambda}(1+\lambda s_t)e^{-\lambda s_t}\right]^{\alpha_g} \\
Y_{st} &= A_{st}(q_t X_{st})^{1-\alpha_s}\left[\theta_\ell(1-e^{-\lambda s_t})\right]^{\alpha_s} \\
m_t &= e^{-\lambda s_t} \\
\delta(u_t) &= (1/\tau)u_t^{\tau}\,, \quad \tau > 1\,.
\end{aligned}$$

The first order conditions (for an interior optimum) are:

$$\begin{aligned}
C_{hgt} &: \quad \lambda_h \gamma_h\, C_{ht}^{1-\sigma}\, C_{hgt}^{-1} = \mu_{1t} \\
C_{hst} &: \quad \lambda_h\,(1-\gamma_h)\; C_{ht}^{1-\sigma}\, C_{hst}^{-1} = \mu_{2t} \\
C_{\ell gt} &: \quad \lambda_\ell \gamma_\ell\, C_{\ell t}^{1-\sigma}\, C_{\ell gt}^{-1} = \mu_{1t} \\
C_{\ell st} &: \quad \lambda_\ell\,(1-\gamma_\ell)\; C_{\ell t}^{1-\sigma}\, C_{\ell st}^{-1} = \mu_{2t} \\
X_{st} &: \quad \mu_{1t} = \mu_{2t}\,(1-\alpha_s)\,Y_{st}\,X_t^{-1} \\
u_t &: \quad \frac{\delta'(u_t)}{q_t}K_t = (1-\alpha_g)\,Y_{gt}\,u_t^{-1} \\
K_{t+1} &: \quad \frac{1}{q_t} = \beta\,\mathbb{E}_t\left(\frac{\mu_{1,t+1}}{\mu_{1,t}}\right)\left((1-\alpha_g)\frac{Y_{g,t+1}}{K_{t+1}} + \frac{1-\delta(u_{t+1})}{q_{t+1}}\right)
\end{aligned}$$

and the optimality condition for the cut-off skill level s_t is

$$
\begin{aligned}
s_t: \qquad 0 &= \lambda_h m_t^{'} U_h(C_{hgt}, C_{hst}) - \lambda_\ell m_t^{'} U_\ell(C_{\ell gt}, C_{\ell st}) \\
&+ \mu_{1,t}\left[\alpha_g Y_{gt} H_t^{-1} H_t^{'} - m_t^{'}\left(C_{hgt} - C_{\ell gt}\right)\right] \\
&+ \mu_{2,t}\left[\alpha_s Y_{st} L_t^{-1} L_t^{'} - m_t^{'}\left(C_{hst} - C_{\ell st}\right)\right],
\end{aligned}
$$

where $m_t^{'}$ denotes dm_t/ds_t, and similarly with the other variables.

In Appendix, I show that the optimality conditions can be reduced to the following system:

$$
\frac{\gamma_h}{1-\gamma_h}\frac{C_{hst}}{C_{hgt}} = \frac{1}{P_{st}} \tag{14}
$$

$$
\frac{\gamma_\ell}{1-\gamma_\ell}\frac{C_{\ell st}}{C_{\ell gt}} = \frac{1}{P_{st}} \tag{15}
$$

$$
\Phi\, P_{st}^{(\gamma_h-\gamma_\ell)(1-\sigma)} = \left(\frac{C_{hgt}}{C_{\ell gt}}\right)^{\sigma} \tag{16}
$$

$$
X_{st} = (1-\alpha_s)\, Y_{st}\, P_{st} \tag{17}
$$

$$
K_t u_t^{\tau} = q_t (1-\alpha_g)\, Y_{gt}\,, \tag{18}
$$

the cut-off equation:

$$
\begin{aligned}
&\sigma\gamma_h^{-1} C_{hgt} + (1-\sigma)\alpha_g Y_{gt}\left[\frac{\lambda s_t e^{\lambda s_t}}{1+\lambda s_t}\right] \\
&= \sigma\gamma_\ell^{-1} C_{\ell gt} + (1-\sigma) P_{st}\alpha_s Y_{st}\left[\frac{1}{1-e^{-\lambda s_t}}\right] \qquad (\sigma \neq 1)
\end{aligned} \tag{19}
$$

and Euler equation:

$$
\begin{aligned}
&P_{st}^{(\gamma_h-1)(1-\sigma)} C_{hgt}^{-\sigma} \\
&= q_t \beta\, \mathbb{E}_t\, P_{s,t+1}^{(\gamma_h-1)(1-\sigma)} C_{hg,t+1}^{-\sigma}\left((1-\alpha_g)\frac{Y_{g,t+1}}{K_{t+1}} + \frac{1-\delta(u_{t+1})}{q_{t+1}}\right),
\end{aligned} \tag{20}
$$

where $\frac{1}{P_{st}} = \frac{\mu_{1t}}{\mu_{2t}}$.

Equations (14) - (20), together with the resource constraints characterize the Pareto solution.

3 Quantitative Experiments

In this section, I first characterize the steady state of the economy and then run a series of experiments to asses the performance of the model under different scenarios.

Steady State

In Appendix, I show that steady state of the economy is characterized in 11 variables $(C_{hg}, C_{hs}, C_{\ell g}, C_{\ell s}, u, K, X_s, Y_g, Y_s, P_s, s)$ by the following 11 equations:

$$\frac{\gamma_h}{1-\gamma_h}\frac{C_{hs}}{C_{hg}} = \frac{1}{P_s} \tag{21}$$

$$\frac{\gamma_\ell}{1-\gamma_\ell}\frac{C_{\ell s}}{C_{\ell g}} = \frac{1}{P_s} \tag{22}$$

$$\Phi P_s^{(\gamma_h-\gamma_\ell)(1-\sigma)} = \left(\frac{C_{hg}}{C_{\ell g}}\right)^{\sigma} \tag{23}$$

$$X_s = (1-\alpha_s)\, P_s Y_s \tag{24}$$

$$u^{\tau} = (1-\alpha_g)\, Y_g\, K^{-1}, \tag{25}$$

$$1/\beta = \left((1-\alpha_g)\frac{Y_g}{K} + 1 - \delta(u)\right) \tag{26}$$

$$Y_g = e^{-\lambda s}C_{hg} + (1-e^{-\lambda s})C_{\ell g} + X_s + \delta(u)K \tag{27}$$

$$Y_s = e^{-\lambda s}C_{hs} + (1-e^{-\lambda s})C_{\ell s} \tag{28}$$

$$Y_g = A_g\big[uK\big]^{1-\alpha_g}\left[\frac{1}{\lambda}(1+\lambda s)e^{-\lambda s}\right]^{\alpha_g} \tag{29}$$

$$Y_s = A_s(X_s)^{1-\alpha_s}\left[\theta_\ell(1-e^{-\lambda s})\right]^{\alpha_s} \tag{30}$$

and steady state condition for cut-off equation:

$$\begin{aligned}&\sigma\gamma_h^{-1}C_{hg} + (1-\sigma)\alpha_g Y_g\left[\frac{\lambda s_t e^{\lambda s}}{1+\lambda s}\right] \\ &= \sigma\gamma_\ell^{-1}C_{\ell g} + (1-\sigma)P_s\alpha_s Y_s\left[\frac{1}{1-e^{-\lambda s}}\right] \qquad (\sigma \neq 1)\end{aligned} \tag{31}$$

Stochastic Structure

In addition to the investment shock q_t^I, I assume that there is another shock effecting the skill distribution parameter λ. I also assume that the parameter of the skill distribution follows an AR(1) process

$$\lambda_t = (1 - \rho_\lambda)\lambda + \rho_\lambda \lambda_{t-1} - q_t^X \, ,$$

where λ is the steady state value of the parameter and q_t^I is the skill shock.

With this specification a positive shock ($q_t^X > 0$) leads to a flatter distribution, i.e. an increase in efficiency units across the population at varying degrees. Figure 3 depicts an example for the evolution of the skill distribution after a positive skill shock. In the figure, the curve attached to the parameter value λ_0 denotes the steady state skill distribution and after the shock the distribution initially shifts down (denoted by the thick dashed lines) and then gradually moves back to its steady state position.

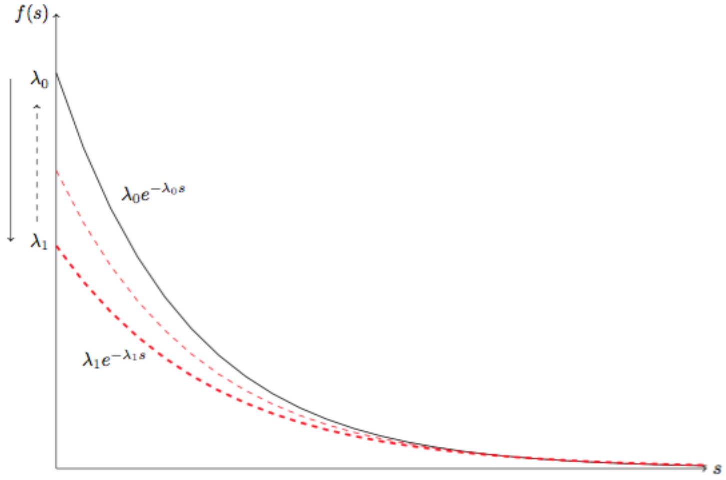


Figure 3: Evolution of A Positive Skill Shock

I assume that the shocks (q_t^I, q_t^X) are jointly driven by the following stochastic process[8]

$$\begin{bmatrix} q_t^I \\ q_t^X \end{bmatrix} = \begin{bmatrix} \rho_I & 0 \\ \chi & \rho_X \end{bmatrix} \begin{bmatrix} q_{t-1}^I \\ q_{t-1}^X \end{bmatrix} + \begin{bmatrix} \nu_t^I \\ \nu_t^X \end{bmatrix} + \begin{bmatrix} 1 \\ a_X \end{bmatrix} \eta_t \,, \qquad |\rho_I|, |\rho_X| < 1 \,,$$

where a^X is a scaling factor.

I also assume that $\nu_t = [\nu_t^I \; \nu_t^X]'$ is a vector white noise process, with the properties

$$E(\nu_t) = 0 \quad \text{for all } t \qquad E(\nu_t \nu_s') = \begin{cases} \Omega & s = t \\ 0 & s \neq t \end{cases}$$

where is the Ω covariance matrix that is assumed to be given by

$$\Omega = \begin{pmatrix} \sigma_I^2 & 0 \\ 0 & \sigma_X^2 \end{pmatrix} .$$

Thus the ν's are both serially and contemporaneously uncorrelated.

The stochastic process also has another innovation term ν_t affecting both investment and skill shocks. I included this additional term just to obtain IRFs to a common shock, rather than IRFs from only the investment or the labor productivity shock separately. In the experiments below, I turn this innovation off unless I am interested specifically in IRFs.

The assumption that the investment and the skill shocks (q_t^I, q_t^X) are driven jointly by the same stochastic process has an intuitive interpretation. When a new technology arrives (an investment shock), initially there is an intensive experimentation around it, either to find the best practice of implementing the technology or to apply the core ideas in different fields.[9] This experimentation stage creates an additional demand for the high skill labor which in terms of the skill distribution given in Figure 3 can be represented as a rightward shift. But then some of the tasks that were being performed by higher skilled agents

[8] Stationary conditions follow from the analysis of the characteristic equation corresponding to the coefficient matrix:

$$\begin{vmatrix} \rho_I - \lambda & 0 \\ \chi & \rho_X - \lambda \end{vmatrix} = 0 \,.$$

The characteristic equation clearly have two solutions $\lambda_1 = \rho_I$ and $\lambda_2 = \rho_X$, and the stationary of the VAR requires $|\rho_I|, |\rho_X| < 1$ as stated above.

[9] See, for example, Jovanovic and Rousseau (2005).

previously will have to be performed by lower skill agents, so the cut off skill level s_t will move to the left. Over time, the experimentation stage reaches maturity and some of the ideas get embodied in the capital stock (can be interpreted as automation), and therefore the whole experimentation process starts to work in the reverse direction until the economy reaches its initial position again. But note that, this initial position (or the steady state of the economy) should be interpreted in relative terms here; the skill distribution is relative to the embodied technology and an agent's position in the distribution is relative to the whole skill distribution.

Results

To test the performance of the model I consider the following scenario that was also used in Karadas (2017): high skill agents work in the capital intensive sector ($\alpha_g < \alpha_s$) and consume capital intensive goods relatively more as well ($\gamma_h > \gamma_\ell$). However, note that utility functions are different than the ones in Karadas (2017); in the latter an agent had the same preferences over the business cycle but the utility functions were non-homothetic, on the other hand, in this paper agents switches between two utility functions depending on whether they work as low or high skill worker. But a similar interpretation goes through in this case as well. Capital (technology) embodies ideas so it is desirable, and higher skill agents work in more capital intensive (sophisticated) jobs and their consumption baskets contain more of these goods. When agents previously working in the service sector switch to work as high skill workers in goods (capital) sector, their consumption baskets changes in a way that low skill agents simply imitate high skill agents' consumption behavior as soon as they start working in goods sector. Therefore, the resulting variation in the average factor content of goods is not due to non-homotheticity as was the case in Karadas (2017), but rather due to the endogenous mobility between the sectors following the technology shock.

For the parameter values, I follow the following procedure. I first choose a set of parameter values for $\alpha_g, \alpha_s, \gamma_h$ and γ_ℓ to the scenario above and then scaling the factor for the service sector A_s is adjusted to ensure that the fraction of high skill agents is 0.5 in the steady state.[10] Finally, parameter values for the capacity utilization (τ) and the measure

[10] This fraction does not need to be 0.5, but for a meaningful comparison among different cases the measure of high skill agents should be kept the same in the steady state.

of EIS ($1/\sigma$) are borrowed from GHH.

To determine the parameter values of the stochastic process I followed a procedure similar to the one in GHH: After matching the key macroeconomic ratios, I adjust the parameter values of the stochastic process ($\sigma_I, \sigma_X, \rho_\lambda, \chi$) to match the standard deviation and the persistency of output in the data. The parameter values and the resulting macro ratios are presented in Table 1.

σ	2.00	σ_I	0.02	C/Y	0.80
β	0.96	σ_X	0.009	I/Y	0.20
γ_ℓ	0.30	σ_ν	0.00	K/Y	1.98
γ_h	0.80	ρ_I	0.00	X_s/Y	0.08
α_g	0.60	ρ_X	0.00	Y_g/Y	0.70
α_s	0.80	ρ_λ	0.60	P_sY_s/Y	0.38
τ	1.42	a_X	0.00	$\mathcal{U}_h/\mathcal{U}_\ell$	1.18
λ	0.30	χ	0.50	m	0.50
λ_h	0.50	A_s	1.80		

Table 1: Paramater Values and Ratios

Table 2 presents the moments of the model. For convenience I also report the corresponding table from GHH paper together with the US data as reported in their paper to compare with my results.

	USA			GHH			Model		
	Std	AR	Corr	Std	AR	Corr	Std	AR	Corr
Output	3.50	0.66	1.00	3.50	0.66	1.00	3.50	0.66	1.00
Consumption	2.20	0.72	0.74	2.20	0.94	0.79	2.34	0.79	0.50
Investment	10.5	0.25	0.68	11.6	0.50	0.90	15.5	0.38	0.85
Hours	2.10	0.39	0.81	2.20	0.66	1.00	3.61	0.60	0.95
Productivity	2.20	0.77	0.82	1.30	0.66	1.00	1.13	0.84	0.07
Capital	–	–	–	5.60	0.99	0.52	5.46	0.98	0.68
Utilization	–	–	–	6.00	0.52	0.61	5.04	0.42	0.42

Table 2: Moments

First, note that the model captures the persistence of consumption in the data much

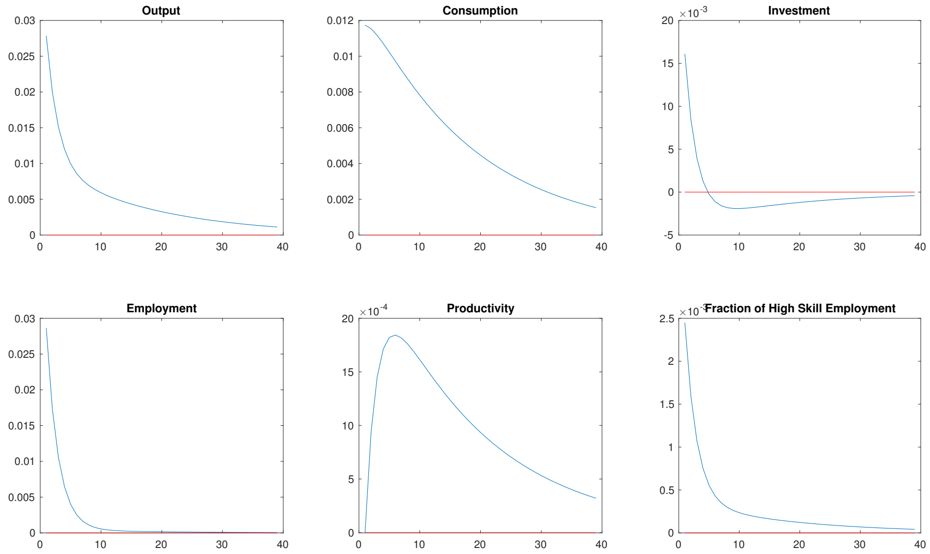


Figure 4: Impulse Response Functions

better than GHH. Persistencies of other variables are very comparable to GHH.

GHH measures the amplification in their model by the ratio of the standard deviation of the innovation to the standard deviation of output, σ_ν/σ_Y. In their model, this ratio is $5.15/3.5 = 1.47$ and the corresponding ratio in my model is $2.0/3.5 = 0.57$, so this model generates significant amplification compare to the GHH model.

Note that the innovation term ν_t^I affects both the investment shock q^I and the evolution of the skill distribution parameter (q^X) through the correlation parameter χ. My computations show that the contribution of the skill shock q^X is much more important. Therefore, the present model suggests that the labor productivity shocks might be the main drivers of business cycles rather than the investment shocks as in GHH.

One of the main challenges for RBC models is that they cannot match the very low co-movement (even slightly negative) between productivity and output in data. But the model presented here can generate very low (almost zero) co-movement between productivity and output. The reason for this is that over the expansion workers with increasingly fewer efficiency units as high skill labor start working in the goods production sector, which directly translates into low productivity. This is a well-known narrative, but hard to incorporate into the standard models. The model presented here works just as in this narrative.

Impulse response functions for the labor productivity that is presented in Figure 4 supports this common narrative that I have just mentioned.

These results suggest that performance of the model is very comparable to standard models even though it does not include any work-leisure trade-off, which is one of the main mechanisms in the standard models. In these models, what stops an agent working more is the foregoing utility from leisure. But this mechanism does not exist here, instead, an agent's position in the skill distribution determines whether she can work more, which can be interpreted in this model as whether the agent can switch to the goods sector.

The model also allows some other interesting numerical exercises. In the rest of this section, I briefly present the results of two such exercises.

For example, two economies that are different only in terms of their skill distributions can be compared in terms of the characteristics of their business cycles. The results of such an exercise are reported in Table 3. The results show that an economy with a flatter skill distribution ($\lambda = 0.25$) exhibits more volatility and persistence overall in comparison to an economy with $\lambda = 0.30$.

	Model ($\lambda = 0.25$)			Model ($\lambda = 0.30$)		
	Std	AR	Corr	Std	AR	Corr
Output	4.14	0.69	1.00	3.52	0.68	1.00
Consumption	2.67	0.83	0.55	2.34	0.79	0.50
Investment	17.10	0.41	0.86	15.54	0.38	0.85
Hours	4.50	0.60	0.95	3.61	0.60	0.95
Productivity	1.37	0.79	-0.10	1.13	0.84	0.07
Capital	6.27	0.98	0.68	5.46	0.98	0.68
Utilization	5.68	0.46	0.42	5.04	0.42	0.42

Table 3: Moments - Two Economies with Different Skill Distributions

As an another exercise, I study how the gap between the preference parameters of high and low skill agents affect the results. For this exercise suppose that in the first economy preference parameters are given as $\gamma_\ell = 0.4$ and $\gamma_h = 0.6$, while in the second economy they are given as $\gamma_\ell = 0.3$ and $\gamma_h = 0.6$, and furthermore suppose that in both economies factor intensities are given as $\alpha_g = 0.8$ and $\alpha_s = 0.6$. The results, given in Table 4, show that the economy on the right in the table which has a wider preference gap, fluctuates slightly more in response to a technology shock.

	Model ($\gamma_\ell = 0.4$, $\gamma_h = 0.6$)			Model ($\gamma_\ell = 0.3$, $\gamma_h = 0.8$)		
	Std	AR	Corr	Std	AR	Corr
Output	3.90	0.65	1.00	4.05	0.66	1.00
Consumption	2.47	0.84	0.65	2.54	0.38	0.83
Investment	27.1	0.37	0.83	29.54	0.38	0.83
Hours	3.47	0.58	0.98	3.63	0.58	0.98
Productivity	0.84	0.96	0.59	0.86	0.97	0.67
Capital	6.27	0.97	0.67	9.43	0.97	0.67
Utilization	6.69	0.65	0.13	5.99	0.64	0.15

Table 4: Moments - Two Economies with Different Preference Gaps

4 Conclusions

In this paper, I studied a simple model in which agents are sorted into two different sectors based on their comparative advantage in the goods sector, and the economy is subject to a pair of shocks to its technology and skill distribution. The model shows that after a positive shock, the fraction of agents working in the goods sector first increases and then gradually drops down after a while. Since those agents who start working in the goods sector replace their service intensive consumption baskets with goods intensive consumption baskets, the demand for goods increases more relative to the service goods along the expansion, which in return results in even more agents being hired in the goods sector. We show that this endogenous mechanism alone can generate sizable amplification and persistence, even without labor-leisure choice, in contrast to the standard models.

All the results are based on the exercises that I carried out basically to test whether the model can yield reasonable moments under some plausible scenarios and to see how they compare to the results from a standard model. However, I have not followed a rigorous calibration procedure when I determined the parameter values for the stochastic process. Although, the results seem to be robust to alternative specifications, more work is needed to carry the model to the data.

Another point that I would like to emphasize is that the model does not claim to be a micro-founded model. The economy is described from the perspective of a modeler (or econometrician) who observes certain patterns in the data and would like to incorporate them into the model as behavioral rules. Therefore, it can be considered as an econometric model with a story behind it, which return imposes some structural relations on the endogenous variables of the model (mostly through the market clearing and the assumptions on the production functions) and the modeler's story is not explosive in the sense that the story covers the whole time period from the moment of the initial shock until the economy eventually reaches its steady state again.

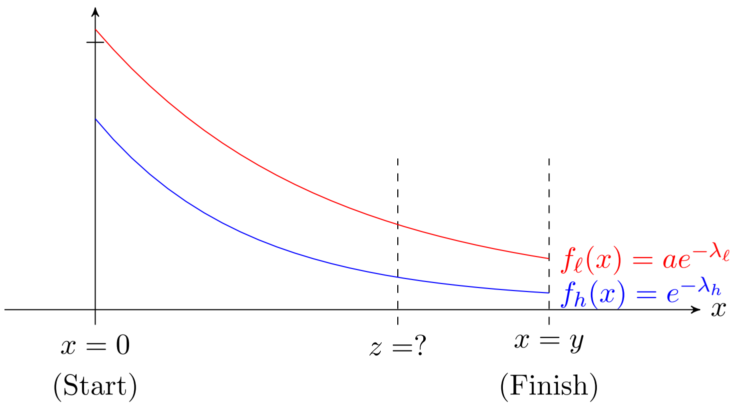
The model proposed is wide open for further development. For example, this model can be extended to include a multidimensional skill distribution in a straightforward manner. As an example, for a two-dimensional skill distribution the following set up can be used: each $i \in [0, 1]$ is endowed with a vector of efficiency units as a high, middle and low skilled worker, denoted by $e_h(i)$, $e_m(i)$ and $e_\ell(i)$, respectively. Furthermore, assume that each individual is endowed with a general skill level $e(i)$ as well. Consequently, the vector $\xi(i) = (e(i), e_\ell(i), e_m(i), e_h(i)) \in \Theta \times \Theta_L \times \Theta_M \times \Theta_H$ completely describes i's skill endowment. Now, if we assume that the general skill level is correlated with the other skills, then there will still be a cut-off rule as in this model.

Finally, I would like to point out that the model can be interpreted and modified as a firm-entry model in which the factor content of demand varies endogenously over the business cycle. For this, consider an endogenous entry model as in Bilbiie, Ghironi and Melitz (2012) in a two sector framework as in this paper. Furthermore, suppose that entering the goods market requires one unit of high-skill labor (sunk-cost) and as soon as an individual starts working as a high-skill worker his consumption basket changes as well. This model would be similar to the one presented in this paper in terms the core mechanism at work. I leave this extension for a future research.

Appendix

A Simple Example of Division of Labor

Consider a production process that requires a certain number of tasks to be performed in order to generate one unit of output. Let us assume there is a continuum of tasks x normalized from 0 to y, that is the production process starts at 0 and generates 1 unit of output when it reaches the point y. Also I assume that tasks are ordered in a descending fashion in terms of difficulty, starting from the most difficult one towards easier tasks. Suppose that there are two types of labor, high-skill (H) and low-skill (L), that can perform any task $x \in [0, y]$, but at different productivities. In particular, I assume that H can process tasks at a rate denoted by $f_h(x)$, $x \in [0, y]$. For example, it takes $\int_0^z f_h(x)\,dx$ units of time to progress until task z, starting from the beginning ($x = 0$). Similarly, low-skill labor have a productivity function denoted by $f_l(x)$.



The firm's problem is to allocate tasks to different types of labor in order to minimize total labor cost. Since more difficult tasks are at the beginning of the production process, these are to be allocated to the high-skill labor; say tasks between 0 and z are performed by H-types, and from z to y, by L-types. Then, the firm's problem is:

$$Q(y) = \min_z \left\{ w_h \int_0^z f_h(x)\,dx + w_h \int_z^y f_l(x)\,dx \right\} \qquad (*)$$

where w_h and w_l are the wage rates of H-, and L-types, respectively, expressed in per unit of time. Here, y is a measure of the size (or scale) of the production, and $Q(y)$ is the minimum cost of producing 1 unit of output for a firm of size y.

For simplicity, I assume that the productivity functions take the exponential form:

$$f_h(x) = e^{-x} \quad \text{and} \quad f_h(x) = ae^{-\lambda x}\,,$$

where I normalize the coefficients of the productivity function for H-types, but I leave two free parameters for the L-types so that both absolute and comparative advantage can arise depending the parameters a and λ.

The first order condition of $(*)$ with respect to z:[11]

$$w_h e^{-z} - aw_\ell e^{-\lambda z} = 0,$$

and solving for z gives the threshold task:

$$z = \frac{1}{1-\lambda} \ln\left(\frac{w_h}{aw_\ell}\right).$$

Putting z back into the cost function $Q(y)$ gives

$$Q(y) = \left(w_h + \frac{aw_\ell}{\lambda} e^{-\lambda y}\right) - \left(\frac{1+\lambda}{\lambda}\right) w_h^{1-\frac{1}{1-\lambda}} w_\ell^{\frac{1}{1-\lambda}}$$

And, from Shepard's lemma, we can derive the demand for high and low skilled labor:

$$D_h = 1 + \left(\frac{1+\lambda}{1-\lambda}\right)\left(\frac{w_h}{aw_\ell}\right)^{-\frac{1}{1-\lambda}}$$

and

$$D_\ell = \frac{a}{\lambda} e^{-\lambda y} + \frac{1}{\lambda}\left(\frac{1+\lambda}{1-\lambda}\right)\left(\frac{w_h}{aw_\ell}\right)^{1-\frac{1}{1-\lambda}}$$

Note that the size of the firm appears only in the demand function of low-skilled labor, and as the size of the firm increases benefits of employing the high-skilled labor goes down, and as a result the fraction of the less-skilled labor rises in total employment. Interestingly, these

[11] Note that with the exponential productivity function the expression in $(*)$ becomes

$$Q(y) = \min_z \left\{ w_h \left(1 - e^{-z}\right) + \frac{aw_\ell}{\lambda}\left(e^{-y\lambda} - e^{-z\lambda}\right)\right\} \quad (**)\,,$$

and this equation also reveals how this problem can be expressed as the weighted sum (convolution) of two exponential random variables.

labor demand functions are very similar to the one obtained from Cobb-Douglas production functions.

Expressions for L_t and H_t

First note that under with the exponential distribution the labor market clearing conditions become

$$
\begin{aligned}
L_t &= \int_0^{s_t} e_\ell(s) f(s)\, ds = \int_0^{s_t} \theta_\ell f(s)\, ds \\
&= \theta_\ell \int_0^{s_t} \lambda e^{-\lambda s}\, ds \\
&= \theta_\ell (1 - e^{-\lambda s_t})
\end{aligned}
$$

and

$$
\begin{aligned}
H_t &= \int_{s_t}^{\infty} e_h(s) f(s)\, ds = \int_{s_t}^{\infty} s f(s)\, ds \\
&= \int_{s_t}^{\infty} \lambda s\, e^{-\lambda s}\, ds \\
&= \frac{1}{\lambda}(1 + \lambda s_t) e^{-\lambda s_t}
\end{aligned}
$$

Derivation of the First Order Conditions

The first order conditions (for an interior optimum) are:

$$
\begin{aligned}
C_{hgt} :& \quad \lambda_h \gamma_h\, C_{ht}^{1-\sigma}\, C_{hgt}^{-1} = \mu_{1t} && (32) \\
C_{hst} :& \quad \lambda_h\, (1 - \gamma_h)\, C_{ht}^{1-\sigma}\, C_{hst}^{-1} = \mu_{2t} && (33) \\
C_{\ell gt} :& \quad \lambda_\ell \gamma_\ell\, C_{\ell t}^{1-\sigma}\, C_{\ell gt}^{-1} = \mu_{1t} && (34) \\
& && (35)
\end{aligned}
$$

$$C_{\ell st} \;:\; \lambda_\ell \left(1-\gamma_\ell\right) C_{\ell t}^{1-\sigma} = \mu_{2t} C_{\ell st} \tag{36}$$

$$X_{st} \;:\; \mu_{1t} = \mu_{2t}\left(1-\alpha_s\right) Y_{st}\, X_t^{-1} \tag{37}$$

$$u_t \;:\; \frac{\delta'(u_t)}{q_t} K_t = \left(1-\alpha_g\right) Y_{gt}\, u_t^{-1} \tag{38}$$

$$K_{t+1} \;:\; \frac{1}{q_t} = \beta \mathbb{E}_t \left(\frac{\mu_{1,t+1}}{\mu_{1,t}}\right)\left(\frac{(1-\alpha_g) Y_{g,t+1}}{K_{t+1}} + \frac{1-\delta(u_{t+1})}{q_{t+1}}\right) \tag{39}$$

and the optimality condition for the cut-off skill level s_t is

$$s_t \;:\; 0 = \lambda_h m_t^{'} U_h(C_{hgt}, C_{hst}) - \lambda_\ell m_t^{'} U_\ell(C_{\ell gt}, C_{\ell st}) \tag{40}$$

$$+ \mu_{1,t}\left[\alpha_g Y_{gt} H_t^{-1} H_t^{'} - m_t^{'}\left(C_{hgt} - C_{\ell gt}\right)\right] \tag{41}$$

$$+ \mu_{2,t}\left[\alpha_s Y_{st} L_t^{-1} L_t^{'} - m_t^{'}\left(C_{hst} - C_{\ell st}\right)\right], \tag{42}$$

where $m_t^{'}$ denotes dm_t/ds_t, and similarly with the other variables.

Rearranging the FOCs, we obtain the following relationships

$$
\begin{aligned}
(32)/(33) \;&:\; \frac{\gamma_h}{1-\gamma_h}\frac{C_{hst}}{C_{hgt}} = \frac{\mu_{1t}}{\mu_{2t}} \\
(34)/(36) \;&:\; \frac{\gamma_\ell}{1-\gamma_\ell}\frac{C_{\ell st}}{C_{\ell gt}} = \frac{\mu_{1t}}{\mu_{2t}} \\
(32) = (34) \;&:\; \lambda_h \gamma_h\, C_{ht}^{1-\sigma}\, C_{hgt}^{-1} = \lambda_\ell \gamma_\ell\, C_{\ell t}^{1-\sigma}\, C_{\ell gt}^{-1} \\
(37) \;&:\; \frac{\mu_{1t}}{\mu_{2t}} = \left(1-\alpha_s\right) Y_{st}\, X_t^{-1} \\
(38) \;&:\; u_t^{\tau} = q_t(1-\alpha_g)\, Y_{gt}\, K_t^{-1} \\
(39) \;&:\; \frac{1}{q_t} = \beta \mathbb{E}_t \left(\frac{\mu_{1,t+1}}{\mu_{1,t}}\right)\left((1-\alpha_g)\frac{Y_{g,t+1}}{K_{t+1}} + \frac{1-\delta(u_{t+1})}{q_{t+1}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \lambda_h U_h(C_{hgt}, C_{hst}) \mu_{1,t}^{-1} - \lambda_\ell U_\ell(C_{\ell gt}, C_{\ell st}) \mu_{1,t}^{-1} \\
&+ \left[\alpha_g Y_{gt} H_t^{-1} H_t^{'} (m_t^{'})^{-1} - \left(C_{hgt} - C_{\ell gt}\right)\right] \\
&+ \frac{\mu_{2t}}{\mu_{1t}}\left[\alpha_s Y_{st} L_t^{-1} L_t^{'} (m_t^{'})^{-1} - \left(C_{hst} - C_{\ell st}\right)\right].
\end{aligned}
$$

Define $\frac{1}{P_{st}} = \frac{\mu_{1t}}{\mu_{2t}}$ and note that

$$\begin{aligned}
\mu_{1t} &= \lambda_h\left(\gamma_h\right) C_{ht}^{1-\sigma} C_{hgt}^{-1} \\
&= \lambda_h\left(\gamma_h\right) \left[\left(\frac{C_{hst}}{C_{hgt}}\right)^{1-\gamma_h} C_{hgt}\right]^{1-\sigma} C_{hgt}^{-1} \\
&= \lambda_h\left(\gamma_h\right) \left[\left(\frac{1-\gamma_h}{\gamma_h}\right)^{1-\gamma_h} \left(\frac{1}{P_{st}}\right)^{1-\gamma_h} C_{hgt}\right]^{1-\sigma} C_{hgt}^{-1} \\
&= \lambda_h\left(\gamma_h\right) \left[\tilde{\gamma}_h \gamma_h^{-1} P_{st}^{\gamma_h-1} C_{hgt}\right]^{1-\sigma} C_{hgt}^{-1} \\
&= \lambda_h \gamma_h^{\sigma}\, \tilde{\gamma}_h^{1-\sigma} P_{st}^{(\gamma_h-1)(1-\sigma)} C_{hgt}^{-\sigma}, \qquad \tilde{\gamma}_h = \gamma_h^{\gamma_h}(1-\gamma_h)^{(1-\gamma_h)}.
\end{aligned}$$

Similarly,

$$\mu_{1t} = \lambda_\ell \gamma_\ell^{\sigma}\, \tilde{\gamma}_\ell^{1-\sigma} P_{st}^{(\gamma_\ell-1)(1-\sigma)} C_{\ell gt}^{-\sigma}, \qquad \tilde{\gamma}_\ell = \gamma_\ell^{\gamma_\ell}(1-\gamma_\ell)^{(1-\gamma_\ell)}.$$

Therefore, we can rewrite the third condition as

$$\begin{aligned}
\lambda_h \gamma_h^{\sigma}\, \tilde{\gamma}_h^{1-\sigma} P_{st}^{(\gamma_h-1)(1-\sigma)} C_{hgt}^{-\sigma} &= \lambda_\ell \gamma_\ell^{\sigma}\, \tilde{\gamma}_\ell^{1-\sigma} P_{st}^{(\gamma_\ell-1)(1-\sigma)} C_{\ell gt}^{-\sigma} \\
\Phi\, P_{st}^{(\gamma_h-1)(1-\sigma)} C_{hgt}^{-\sigma} &= P_{st}^{(\gamma_\ell-1)(1-\sigma)} C_{\ell gt}^{-\sigma} \\
\Phi P_{st}^{(\gamma_h-\gamma_\ell)(1-\sigma)} C_{hgt}^{-\sigma} &= C_{\ell gt}^{-\sigma} \\
\Phi\, P_{st}^{(\gamma_h-\gamma_\ell)(1-\sigma)} &= \left(\frac{C_{hgt}}{C_{\ell gt}}\right)^{\sigma}
\end{aligned}$$

where

$$\Phi = \frac{\lambda_h \gamma_h^{\sigma}\, \tilde{\gamma}_h^{1-\sigma}}{\lambda_\ell \gamma_\ell^{\sigma}\, \tilde{\gamma}_\ell^{1-\sigma}}$$

$$
\begin{aligned}
0 =&\lambda_h U_h(C_{hgt}, C_{hst})\mu_{1,t}^{-1} - \lambda_\ell U_\ell(C_{\ell gt}, C_{\ell st})\mu_{1,t}^{-1} \\
&+ \left[\alpha_g Y_{gt} H_t^{-1} H_t^{'}(m_t^{'})^{-1} - (C_{hgt} - C_{\ell gt})\right] + P_{st}\left[\alpha_s Y_{st} L_t^{-1} L_t^{'}(m_t^{'})^{-1} - (C_{hst} - C_{\ell st})\right] \\
=&\lambda_h \frac{1}{1-\sigma} C_{ht}^{1-\sigma} \lambda_h^{-1} \gamma_h^{-1} C_{ht}^{-1+\sigma} C_{hgt} - \lambda_\ell U_\ell(C_{\ell gt}, C_{\ell st})\mu_{1,t}^{-1} \\
&+ \left[\alpha_g Y_{gt} H_t^{-1} H_t^{'}(m_t^{'})^{-1} - (C_{hgt} - C_{\ell gt})\right] + P_{st}\left[\alpha_s Y_{st} L_t^{-1} L_t^{'}(m_t^{'})^{-1} - (C_{hst} - C_{\ell st})\right] \\
=&\frac{1}{1-\sigma}\left[\gamma_h^{-1} C_{hgt} - \gamma_\ell^{-1} C_{\ell gt}\right] \\
&+ \left[\alpha_g Y_{gt} H_t^{-1} H_t^{'}(m_t^{'})^{-1} - (C_{hgt} - C_{\ell gt})\right] + P_{st}\left[\alpha_s Y_{st} L_t^{-1} L_t^{'}(m_t^{'})^{-1} - (C_{hst} - C_{\ell st})\right] \\
=&\frac{1}{1-\sigma}\left[\gamma_h^{-1} C_{hgt} - (1-\sigma)\big[C_{hgt} + P_{st} C_{hst}\big]\right] + \alpha_g Y_{gt} H_t^{-1} H_t^{'}(m_t^{'})^{-1} \\
&- \frac{1}{1-\sigma}\left[\gamma_\ell^{-1} C_{\ell gt} - (1-\sigma)\big[C_{\ell gt} + P_{st} C_{\ell st}\big]\right] + P_{st}\alpha_s Y_{st} L_t^{-1} L_t^{'}(m_t^{'})^{-1} \\
=&\frac{1}{1-\sigma}\big[\sigma\gamma_h^{-1} C_{hgt}\big] + \alpha_g Y_{gt} H_t^{-1} H_t^{'}(m_t^{'})^{-1} \\
-&\frac{1}{1-\sigma}\big[\sigma\gamma_\ell^{-1} C_{\ell gt}\big] + P_{st}\alpha_s Y_{st} L_t^{-1} L_t^{'}(m_t^{'})^{-1}
\end{aligned}
$$

Recall that $m_t = e^{-\lambda s_t}$ to compute $m_t^{'} = -\lambda e^{-\lambda s_t}$ and other relevant expressions

$$
\begin{aligned}
&L_t = \theta_\ell(1 - e^{-\lambda s_t}) && L_t^{'} = \theta_\ell \lambda e^{-\lambda s_t} && L_t^{'}(m_t^{'})^{-1} = -\theta_\ell && L_t^{-1} L_t^{'}(m_t^{'})^{-1} = \frac{-1}{1 - e^{-\lambda s_t}} \\
&H_t = \frac{1}{\lambda}(1 + \lambda s_t) e^{-\lambda s_t} && H_t^{'} = -\lambda s_t e^{-\lambda s_t} && H_t^{'}(m_t^{'})^{-1} = s_t && H_t^{-1} H_t^{'}(m_t^{'})^{-1} = \frac{\lambda s_t e^{\lambda s_t}}{1 + \lambda s_t}
\end{aligned}
$$

Hence, we find

$$
\begin{aligned}
&\sigma\gamma_h^{-1} C_{hgt} + (1-\sigma)\alpha_g Y_{gt}\left[\frac{\lambda s_t e^{\lambda s_t}}{1 + \lambda s_t}\right] \\
&= \sigma\gamma_\ell^{-1} C_{\ell gt} + (1-\sigma) P_{st}\alpha_s Y_{st}\left[\frac{1}{1 - e^{-\lambda s_t}}\right] \qquad (\sigma \neq 1)
\end{aligned}
$$

Now we can rewrite the FOCs as

$$\frac{\gamma_h}{1-\gamma_h}\frac{C_{hst}}{C_{hgt}} = \frac{1}{P_{st}} \tag{43}$$

$$\frac{\gamma_\ell}{1-\gamma_\ell}\frac{C_{\ell st}}{C_{\ell gt}} = \frac{1}{P_{st}} \tag{44}$$

$$\Phi\, P_{st}^{(\gamma_h-\gamma_\ell)(1-\sigma)} = \left(\frac{C_{hgt}}{C_{\ell gt}}\right)^{\sigma} \tag{45}$$

$$X_{st} = (1-\alpha_s)\, Y_{st}\, P_{st} \tag{46}$$

$$K_t u_t^{\tau} = q_t(1-\alpha_g)\, Y_{gt} \tag{47}$$

cut-off equation as

$$\begin{aligned} &\sigma\gamma_h^{-1} C_{hgt} + (1-\sigma)\alpha_g Y_{gt}\left[\frac{\lambda s_t e^{\lambda s_t}}{1+\lambda s_t}\right] \\ &= \sigma\gamma_\ell^{-1} C_{\ell gt} + (1-\sigma) P_{st}\alpha_s Y_{st}\left[\frac{1}{1-e^{-\lambda s_t}}\right] \qquad (\sigma \neq 1) \end{aligned} \tag{48}$$

and Euler equation as

$$\begin{aligned} &P_{st}^{(\gamma_h-1)(1-\sigma)} C_{hgt}^{-\sigma} \\ &= q_t \beta\, \mathbb{E}_t\, P_{s,t+1}^{(\gamma_h-1)(1-\sigma)} C_{hg,t+1}^{-\sigma}\left((1-\alpha_g)\frac{Y_{g,t+1}}{K_{t+1}} + \frac{1-\delta(u_{t+1})}{q_{t+1}}\right). \end{aligned} \tag{49}$$

Equations (43) - (49), together with the resource constraints characterize the Pareto solution.

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