Ch. 3 - Basic Principles of Counting and Probability

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Summer, 2016

Motivation

- Calculations in probability theory often involve working out the number of different ways in which something can happen.
- Since simply listing the ways can be very tedious (and often unreliable), it is helpful to work out some techniques for doing this kind of counting.
- Suppose from a population of size 1000 ($= N$) we would like to chose a sample of size 30 ($= n$) for each of ten trials.
- One might be interested in the following question: *how many different samples of size n are possible from a population of size N?*
  - Think about this: is it possible to generate 5 different samples of size 3 from a population of size $N = 4$?
  - What about if $N = 5$?

- Let us give an example: suppose that the population size is 10 ($N = 10$) and we would like to determine the number of possible samples of size 3 ($n = 3$). How many different samples can be generated, without replacement?
  - Take $N = \{1, 2, \ldots, 10\}$
  - Let us denote the realized sample by S

$$S = \{n_1, n_2, n_3\}$$

    then the question is how many such S are possible?
  - n_1 can be one of the ten population members, and n_2 can be one of the remaining 9 population members, and finally n_3 can be one of the remaining 8 population members.
  - Therefore,

$$\# \text{ of } \textbf{ordered} \text{ triples } (n_1, n_2, n_3) = 10 \times 9 \times 8$$

  - But this is not exactly what we wanted because S should not care the order of its members. For example,

$$\{2, 5, 8\} \quad \text{and} \quad \{8, 5, 2\}$$

    are the same sample, but they were counted separately in the previous expression.

- Therefore, we need to adjust the formula to take into account these double counting.
- Let us continue with the same example: we said $\{2, 5, 8\}$ and $\{8, 5, 2\}$ are the same thing as a sample.
- Let us list all the other same things that we can construct by using the same three members $\{2, 5, 8\}$:

n_1	n_2	n_3
2	5	8
2	8	5
5	2	8
5	8	2
8	2	5
8	5	2

- Therefore, the same three elements, $\{2, 5, 8\}$, appear 6 times in the formula $10 \times 9 \times 8$ with different orderings.

- And of course this is true for all the other triples, therefore

$$\# \text{ of (non-repeating) triples } (\{n_1, n_2, n_3\}) = \frac{10 \times 9 \times 8}{6} = 1200$$

- This is the number of different samples of size 3 that can be drawn from a population of size 10.
- Let us focus a little more on how we can think about the denominator in the expression above
- Basically, each $\{n_1, n_2, n_3\}$ triple can be written as 6 different **ordered** triples, because
  - there are three spots, $| - | - | - |$, to be filled with these three elements
  - the first spot can be filled with one of the three elements $\rightarrow$ in 3 different ways
  - the second spot can be filled with one of the remaining two elements
    $\rightarrow$ in 2 different ways
  - the third spot can be filled with the only remaining element $\rightarrow$ in 1 different way
  - Therefore, there are

    $$3 \times 2 \times 1 = 6$$

    ordered triples

Counting Formulas

Example:

How many different samples of size 4 can be generated from a population of size 13?

-
  $$\# \text{ of } \textbf{ordered} \text{ quadruples } (n_1, n_2, n_3, n_4) = 13 \times 12 \times 11 \times 10$$
-
  $$\# \text{ of double-counting for each quadruple } = 4 \times 3 \times 2 \times 1$$
- Therefore, the number of different samples is equal to
  $$\# \text{ of (non-repeating) quadruples } = \frac{13 \times 12 \times 11 \times 10}{4 \times 3 \times 2 \times 1} = 715$$
- Next, we are going to generalize the ideas we have used in these computations...

Suppose we have x distinguishable objects.

- **Number of Orderings:** The total number of possible ways of ordering these x objects is given by

$$x(x-1)(x-2)\dots 2.1$$

  which is denoted by $x!$, i.e.

$$x! = x(x-1)(x-2)\dots(2)(1)$$

  → To find the number of ways of doing something, multiply the number of choices available at each stage.

- **Permutations (Arrangements):** The total number of permutations of x objects chosen from n objects is

$$n(n-1)(n-2)\dots(n-x+1)$$

  - Note that in terms of factorials this can be expressed simply as

$$\frac{n!}{(n-x)!}.$$

  - And we are going to use the following notation for permutations:

$$P_x^n = \frac{n!}{(n-x)!}$$

- **Combinations:** The total number of combinations of x objects chosen from n objects is

$$C_x^n = \frac{n!}{(n-x)! \cdot x!}$$

- Note that C_x^n is just P_x^n divided by $x!$, i.e.

$$C_x^n = \frac{1}{x!} P_x^n$$

- Going back to the first example, with these definitions,
  - # of ordered triples $\longrightarrow P_3^{10}$
  - # of (non-repeating) triples $\longrightarrow C_3^{10} = \frac{1}{3!} P_3^{10}$

Example 1

How many four-letter code words can be formed using a standard 26-letter alphabet

1. if repetition is allowed?

   Four spots

   $$|-|-|-|-|$$

   to be filled, each with 26 possibilities, so there are $26 \times 26 \times 26 \times 26 = 26^4$ *four-letter codes*

2. if repetition is not allowed?

   $P_4^{26} = \frac{26!}{22!} = 26 \times 25 \times 24 \times 23$

3. if no two code words should contain the same set of four letters (i.e. at least one letter should be different any two pairs of code words)
   $C_4^{26} = \frac{26!}{22! \times 4!} = 26 \times 25 \times 23$

Example 2

Seven of Miss Murphys students are girls and five are boys.

1. Miss Murphy wants to seat 12 of her students in a row for a class picture. How many different seating arrangements are there?

   Ans. 12!

2. In how many different ways can she seat the 7 girls together on the left, and then the 5 boys together on the right?

   Ans. $7! \times 5!$

Example 3

The Library of Science Book Club offers three books from a list of 42. If you circle three choices from a list of 42 numbers on a postcard, how many possible choices are there?

Ans. C_3^{42}

Example 4

Given a class of 12 girls and 10 boys.

1. In how many ways can a committee of five consisting of 3 girls and 2 boys be chosen?

   Ans. First note that the order of the children in the committee does not matter. From 12 girls we can choose C_3^{12} different groups of three girls. From the 10 boys we can choose C_2^{10} different groups. Thus, the total number of committee is $C_3^{12}.C_2^{10}$

2. How many of the possible committees of five have no boys?(i.e. consists only of girls)

   Ans. $C_0^{10}.C_5^{12}$

Example 5.

How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of 3 faculty members from the mathematics department and 4 from the computer science department, if there are 9 faculty members of the math department and 11 of the CS department?

Ans. $C_3^9 . C_4^{11}$

Sample Space:

The set of all possible outcomes from an experiment is called the sample space of the experiment, denoted by S.

Examples

1. If the experiment consists of tossing a coin, there are two possible outcomes in this experiment: heads and tails:

$$S = \{Heads(H), Tails(T)\}$$

2. If the experiment consists of observing the reported SAT scores of a randomly selected student at NYU, the sample space would be the set of positive integers between 200 and 800 that are multiples of ten:

$$S = \{200, 210, 220, \ldots, 790, 800\}$$

3. What is the sample space if toss a coin twice? In this case the sample space is:

$$S = \{HH, HT, TH, TT\}$$

- Think of the last example, what kind of outcomes you might be interested in?
- Obviously you might be interested in any of four possible outcomes.
- But you might also be interested in probability of observing at least one Heads.
  - In that case you need to consider not just a single possible outcome but *a collection of possible outcomes*.
  - Therefore if you are interested in finding the probability of observing at least one Heads in this experiment, you need to consider the set $\{HH, HT, TH\}$.
  - This motivates the following fundamental definition in probability.

Basic Outcome:

A possible outcome of a random experiment.

In another words, if the sample space is

$$S = \{O_1, O_2, \ldots, O_n\}$$

each O_k is a basic outcome.

Event:

Any collection of possible outcomes of an experiment constitute an event.

In other words, any subset of the sample, S, is called an event. For example, some events

$$\begin{aligned}
E_1 &= \{O_1, O_2, \ldots, O_n\} = S \\
E_2 &= \{O_1, O_2\} \\
E_3 &= \{O_1, O_n\} \\
E_3 &= \{O_3, O_7, O_{11}\}
\end{aligned}$$

Examples cont'd

1. If the sample space is
$$S = \{H, T\}$$
then some events:
$$E_1 = \{H\}$$
$$E_2 = \{T\}$$
$$E_3 = \{H, T\}$$

2. Suppose that somehow we are interested in the following situation: What is the probability of an arbitrary chosen student in NYU to have a SAT score above 600.
In this case, the sample space is
$$S = \{200, 210, 220, \dots, 790, 800\}$$
and the event that we are interested in is
$$\begin{aligned} E_1 &= \{\text{All SAT scores above 600}\} \\ &= \{600, 610, \dots, 790, 800\} \end{aligned}$$

Examples cont'd

Another event that we might be interested in would be:

$$E_2 = \{\text{ SAT scores between 600 and 700}\}$$
$$= \{600, 610, \ldots, 690, 700\}$$

3. As we discussed above if we are interested in finding the probability of observing at least one Heads in this experiment, we need to consider the event:

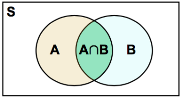
$$E_1 = \{\text{At least one Heads occurs in this experiment}\}$$
$$= \{HH, HT, TH\}$$

Some other events:

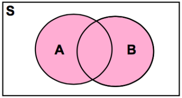
$$E_2 = \{\text{Exactly two Heads occurs}\}$$
$$= \{HH\}$$
$$E_3 = \{\text{In the first toss Heads, and in the second toss Tails comes}\}$$
$$= \{HT\}$$

Generating new events

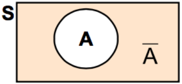
- Intersection of Events: If A and B are two events in a sample space S, then the intersection, $A \cap B$, is the set of all outcomes in S that belong to both A and B



- Union of Events: If A and B are two events in a sample space S, then the union, $A \cup B$, is the set of all outcomes in S that belong to either A or B

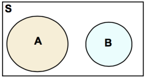


- The Complement of an event A is the set of all basic outcomes in the sample space that do not belong to A. The complement is denoted by $\overline{A}$



Some Definitions

- A and B are Mutually Exclusive events if they have no basic outcomes in common



Formally, two events, A and B, are mutually exclusive if

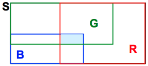
$$A \cap B = \emptyset.$$

- Events $E_1, E_2, \ldots, E_k$ are Collectively Exhaustive events if

$$E_1 \cup E_2 \cup \ldots \cup E_k = S$$

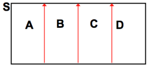
i.e., the events completely cover the sample space

- Consider the following example:



As $B \cup G \cup R = S$, these three events are collectively exhaustive. However, they are not mutually exclusive (their pairwise intersections is not empty).

- Consider another example:



As $A \cup B \cup C \cup D = S$, these four events are collectively exhaustive and mutually exclusive.

- Events $E_1, E_2, \ldots, E_k$ are said to form a Partition of the sample space if they are both collectively exhaustive and pairwise mutually exclusive.
- In the previous example, A, B, C and D form a partition of the sample space (they are collectively exhaustive and pairwise disjoint, check!).
- Note that for any event A, since $A \cup \overline{A} = S$ and $A \cap \overline{A} = \emptyset$ always hold, A and $\overline{A}$ together form a partition of the sample space.

Example

Experiment: Rolling a die once. Then

- Sample space
$$S = \{1, 2, 3, 4, 5, 6\}$$
- Event that the number rolled is even
$$A = \{2, 4, 6\}$$
- Event that the number rolled is at least 4
$$B = \{4, 5, 6\}$$
- Complement: $\overline{A} = \{1, 3, 5\}$, $\overline{B} = \{1, 2, 3\}$
- Intersection: $A \cap B = \{4, 6\}$
- Union: $A \cup B = \{2, 4, 5, 6\}$
- Are they mutually exclusive? No
- Are they collectively exhaustive? No

Example

There are three red chips and two blue chips in a bowl. The red chips are numbered 1, 2, and 3, respectively, and the blue chips are numbered 1 and 2, respectively. Suppose two chips are to be drawn at random.

1. The sample space for this experiment.

$$S = \{R_1R_2, R_1R_3, R_2R_3, B_1B_2, R_1B_1, R_1B_2, R_2B_1, R_2B_2, R_3B_1, R_3B_2\}$$

2. Write down the event that two chips have the same number.

$$E_1 = \{R_1B_1, R_2B_2\}$$

3. Write down the event that two chips have the same color.

$$E_2 = \{R_1R_2, R_1R_3, R_2R_3, B_1B_2\}$$

4. Find the event that two chips have either the same number or the same color.

$$E_3 = \{R_1R_2, R_1R_3, R_2R_3, B_1B_2, R_1B_1, R_2B_2\}$$

Probability Postulates:

Probability:

The chance that an uncertain event will occur.

Probability Postulates:

- $0 \leq P(A) \leq 1$
- If $A = \cup_{i=1}^{n} O_i$, then

$$P(A) = \sum_{i=1}^{n} P(O_i)$$

  where O_i's are basic outcomes
- $P(S) = 1$

Basic Probability Rules:

- The Complement rule:

$$P(\overline{A}) = 1 - P(A)$$

- The Addition rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- For any two events A and B, we can summarize probabilities and joint probabilities as in the following table

	B	$\overline{B}$	
A	$P(A \cap B)$	$P(A \cap \overline{B})$	$P(A)$
$\overline{A}$	$P(\overline{A} \cap B)$	$P(\overline{A} \cap \overline{B})$	$P(\overline{A})$
	$P(B)$	$P(\overline{B})$	$P(S) = 1$

Example

- Consider a standard deck of 52 cards, with four suits:
- Let event A = card is an Ace
- Let event B = card is from the red suite

Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

$$P(\text{Red} \cup \text{Ace}) = P(\text{Red}) + P(\text{Ace}) - P(\text{Red} \cap \text{Ace})$$

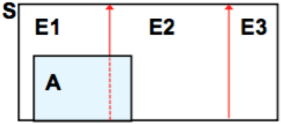
$$= 26/52 + 4/52 - 2/52 = 28/52$$

Don't count the two red aces twice!

Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

Conditional Probability

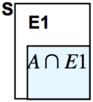
- Consider the following example:
  - In our stat class students are either freshmen (E_1), juniors (E_2), or sophomores (E_3).
  - Some of the freshmen and juniors take Econ 101- the principles (say event A).
  - Also suppose that it is more likely that a freshman taking the principles.
- We can represent this example as



- We could be interested in the following question: *what is the probability that a randomly chosen student in the class takes Econ 101?*
- If you would like to think this question in terms of areas, this probability is equal to

$$\frac{\text{area of A}}{\text{area of S}} = P(A)$$

- Now suppose you are told that the randomly chosen student is a freshman (E_1 occurred), would you change your probability assessment regarding A?
- Continuing with the geometric approach above, knowing E_1 has occurred is equivalent to cutting out and ignoring everything outside the region E_1:



- Importantly, note that we have also cut out the part of A that remained in E_2, and we can express the relevant area as $A \cap E_1$.
- Therefore, knowing that the randomly chosen student is a freshman, the probability that this student also takes Econ 101 is equal to

$$\frac{\text{area of } A \cap E_1}{\text{area of } E_1} = \text{ Probability of } A \text{ given } E_1 \text{ has occurred}$$

- This is the concept of conditional probability. Next, we are going to formalize these ideas.

Conditional Probability: the probability of one event, given that another event has occurred.

- The conditional probability of A given that B has occurred

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- Similarly, the conditional probability of B given that A has occurred

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

Example

Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a CD player (CD). 20% of the cars have both.

What is the probability that a car has a CD player, given that it has AC? i.e., we want to find $P(CD|AC)$

Let us first summarize what we know in the following table

Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a CD player (CD). 20% of the cars have both.

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

Given AC, we only consider the top row (70% of the cars). Of these, 20% have a CD player. 20% of 70% is 28.57%.

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

$$P(CD \mid AC) = \frac{P(CD \cap AC)}{P(AC)} = \frac{.2}{.7} = .2857$$

Multiplication Rule

- Multiplication Rule for two events A and B:

$$P(A \cap B) = P(A|B) \cdot P(B)$$

- Also

$$P(A \cap B) = P(B|A) \cdot P(A)$$

- Basically, the multiplication rule tells us how to calculate $P(A \cap B)$ when we know either $P(A|B)$ or $P(B|A)$.
- Note that the multiplication rules can be obtained simply by rearranging the conditional probability rules.

Example

$$P(Red \cap Ace) = P(Red|Ace) \cdot P(Ace) = \left(\frac{2}{4}\right) \cdot \left(\frac{4}{52}\right) = \frac{2}{52}$$

Example: Forgetful Traveler

A tourist wants to visit three capitals A, B, C. He travels first to one capital chosen at random. If he selects A, he next time chooses between B and C with the same probability. If he then selects B, he next time chooses between A and C with the same probability. So he forgets that he had visited A before.

- What is the probability that he will visit all three capitals in three visits?
- What is the probability that he will visit all three capitals in four visits?

Statistical Independence

- Suppose that we have calculated $P(A)$ and $P(A|B)$, and they are equal, i.e.

$$P(A) = P(A|B)$$

  - LHS is the unconditional probability of A, i.e. *without* knowing whether B has occurred or not.
  - RHS is the conditional probability of A, i.e. probability of A *given* that B has already occurred.
  - If the above equality holds, it means that occurrence of B does NOT effect whether A will occur or not, and in that case, we say these two events are independent.
- Two events are statistically independent if

$$P(A) = P(A|B) \quad \text{or} \quad P(B) = P(B|A)$$

Statistical Independence

- We can obtain an alternative definition of statistical independence:
  - If A and B are independent, we know that $P(A) = P(A|B)$.
  - Substitute this into the conditional probability formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \implies P(A) = \frac{P(A \cap B)}{P(B)}$$

  - Cross-multiplying gives us the following alternative condition for checking whether two events are statistically independent.

- Two events are statistically independent if :

$$P(A \cap B) = P(A) \cdot P(B)$$

Events A and B are independent when the probability of one event is not affected by the other event

Consider the previous example:

Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a CD player (CD). 20% of the cars have both.

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

Are the events AC and CD statistically independent?

- We need to check whether $P(AC \cap CD) = P(AC) \cdot P(CD)$?
- But

$$0.2 = P(AC \cap CD) \neq P(AC) \cdot P(CD) = 0.28$$

- So these two events are not statistically independent

Review Example

In a campus restaurant it was found that 35% of all customers order vegetarian meals, and that 50% of all customers are students. Further, 25% of all customers who are students order vegetarian meals.

1. Define the events and probabilities given in the problem.

   V: customer orders a vegetarian meal
   S: customer is a student.
   $P(V) = 0.35$
   $P(S) = 0.50$
   $P(V|S) = 0.25$

2. What is probability that a randomly chosen customer both is a student and orders a vegetarian meal?

   $P(V \cap S) = P(V|S)P(S) = (0.25)(0.50) = 0.125$
   (from the multiplication rule)

3. If a randomly chosen customer orders a vegetarian meal. what is the probability that the customer is student?

$$P(V|S) = \frac{P(V \cap S)}{P(V)} = \frac{0.125}{0.35} = 0.357$$

4. What is the probability that a randomly chosen customer both doesn't order a vegetarian meal and is not a student?

$$\begin{aligned} P(\bar{V} \cap \bar{S}) &= 1 - P(V \cup S) \quad \text{(Draw a venn diagram to see this)} \\ &= 1 - [P(V) + P(S) - P(V \cap S)] \\ &= 1 - [0.35 + 0.50 - 0.125] \\ &= 1 - 0.725 = 0.275 \end{aligned}$$

5. Are events "customers order vegetarian meals" and "customer is a student" independent?

   Since $P(V \cap S) = 0.125 \neq P(V)P(S) = 0.35.0.50 = 0.175$, events V and S are not independent.

6. Are the events "customers order vegetarian meals" and "customer is a student" mutually exclusive?

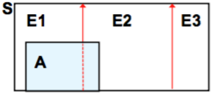
   Since their intersection is not empty (because $P(V \cap S) = 0.125 \neq 0$), these two events are not mutually exclusive.

7. Are events "customers order vegetarian meals" and "customer is a student" collectively exhaustive?

   If these two events were collectively exhaustive then their union would be the all customers, which would in return imply $P(V \cup S) = 1$, but $P(V \cup S) = 0.725$, so they are not collectively exhaustive events.

Bayes' Theorem

- Consider the example that we used to motivate conditional probability:



- There, we discussed how our probability assessment regarding the event A has changed once we learned that E_1 *had already occurred*.
- But the opposite is also possible: sometimes we know that *A has already occurred*, but we do not know which partition of the sample space it is coming from.
- Again assuming that the probability of each event in the picture is proportional to its area, can we say

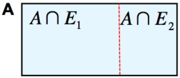
$$P(E_2) > P(E_1)?$$

Yes!

- Now, suppose you learned that A has already occurred, would you change your probability assessment regarding E_1 and E_2?

  $\rightarrow$ Do you still think $P(E_2) > P(E_1)$?

- Knowing that A has occurred we know that we are in the blue region, but we still do not know whether in $A \cap E_1$ or in $A \cap E_2$,



- Now, we can express the probability that the draw is coming from E_1 given that A has already occured as

$$P(E_1|A) = \frac{P(A \cap E_1)}{P(A)}$$
$$= \frac{P(A \cap E_1)}{P(A \cap E_1) + P(A \cap E_2)}$$
$$= \frac{P(A|E_1)P(E_1)}{P(A|E_1)P(E_1) + P(A|E_2)P(E_2)}$$

- Just by looking at the picture, we can say

$$P(E_1|A) > P(E_2|A)$$

Recall that without knowing whether A has occurred, we had concluded exactly the opposite, i.e. $P(E_1) < P(E_2)$:

  - Without knowing anything about the randomly chosen student (whether it takes Econ 101), it is more likely that the student is a junior, but once we know that the student takes Econ 101, it is more likely that the student is a freshman.

- Summary:
  - We started with an initial probability assessment for each element of the partitions

$$P(E_1), \quad P(E_2), \quad \text{and} \quad P(E_3)$$

These are called unconditional probabilities.

  - Then we learned that some related event (A) has occurred
  - and using this new piece of information we updated our initial probability assessments to

$$P(E_1|A), \quad P(E_2|A), \quad \text{and} \quad P(E_3|A)$$

  - This is the content of Bayes's theorem that we will state next.

Bayes' Theorem

$$P(E_i|A) = \frac{P(A|E_i)P(E_i)}{P(A|E_1)P(E_1) + P(A|E_2)P(E_2) + P(A|E_3)P(E_3)}$$

where
$E_i = i^{th}$ element of the partition of the sample space
$A =$ an event that might impact $P(E_i)$

Example

A drilling company has estimated a 40% chance of striking oil for their new well. A detailed test has been scheduled for more information. Historically, 60% of successful wells have had detailed tests, and 20% of unsuccessful wells have had detailed tests. Given that this well has been scheduled for a detailed test, what is the probability that the well will be successful?

Solution 1.

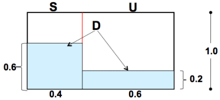
- Let S = successful well and U = unsuccessful well
- $P(S) = 0.4$ and $P(U) = 0.6$ (prior probabilities)
- Define the detailed test event as D
- Then conditional probabilities: $P(D|S) = 0.6$, $P(D|U) = 0.2$
- Goal is to find $P(S|D)$

$$P(S|D) = \frac{P(D|S)P(S)}{P(D|S)P(S) + P(D|U)P(U)} = \frac{(.6)(.4)}{(.6)(.4) + (.2)(.6)} = 0.667$$

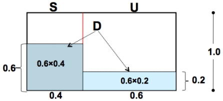
So the revised the probability of success (from the original estimate of .4), given that this well has been scheduled for a detailed test, is .667

Solution 2 - A Geometric Approach.

- In this approach, first we summarize all the information given in the problem using rectangles



- Probabilities are proportional to the respective areas.
- Using this representation we can rephrase the question as: knowing that the draw is coming from one of the blue regions, what is the probability that it is from the one on the left (colored dark below)?



- Then we just need to calculate the ratio, $\frac{\text{Dark Blue Area}}{\text{Total Blue Area}}$, which is

$$\frac{(.6)(.4)}{(.6)(.4) + (.2)(.6)} = 0.667$$

Example

Three plants C_1, C_2, and C_3, produce respectively, $10\%, 50\%$, and 40% of a company's output. Although plant C_1 is small, its manager believes in high quality and 1% of its products are defective. The other two, C_2, and C_3 are worse and produce items that are 3% and 4% defective, respectively. All products are sent to a central warehouse. One item is selected at random and observed to be defective. What is the probability that the defective item is coming from the first plant?

Solution 1.

We are given: $P(C_1) = 0.1, \quad P(C_2) = 0.5, \quad P(C_3) = 0.4$

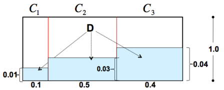
Let us name the event "being defective" as D, then conditional probabilities are: $P(D|C_1) = 0.01$, $P(D|C_2) = 0.03$, $P(D|C_3) = 0.04$

We are asked to find $P(C_1|D)$? Using Bayes' rule:

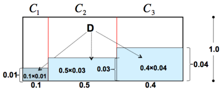
$$
\begin{aligned}
P(C_1|D) &= \frac{P(C_1 \cap C)}{P(D)} \\
&= \frac{P(D|C_1)P(C_1)}{P(D|C_1)P(C_1) + P(D|C_2)P(C_2) + P(D|C_3)P(C_3)} \\
&= \frac{(0.10)(0.01)}{(0.10)(0.01) + (0.5)(0.03) + (0.4)(0.04)} = \frac{1}{32}
\end{aligned}
$$

Solution 2 - A Geometric Approach.

- Summarize the information using rectangles



- Using this representation we can rephrase the question as: knowing that the draw is coming from one of the blue regions, what is the probability that it is from the first segment on the left (colored dark below)?



- Then again we just need to calculate the ratio, $\frac{\text{Dark Blue Area}}{\text{Total Blue Area}}$, which is

$$\frac{(0.10)(0.01)}{(0.10)(0.01)+(0.5)(0.03)+(0.4)(0.04)} = \frac{1}{32}$$