Midterm 1 - Solutions

Statistics - NYU, Summer 2016
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[1] A college library has five copies of a certain text on reserve. Two copies (1 and 2) are first printings, and the other three (3, 4, and 5) are second printings. Consider the following experiment: *A student examines these books in random order, stopping only when a second printing has been selected.*

a) List the outcomes in the sample space. (5pt)

$$S = \{3, 4, 5, 13, 14, 15, 23, 24, 25, 123, 124, 125, 213, 214, 215\}$$

b) Let A denote the event that exactly one book must be examined. What outcomes are in A? (5pt)

$$A = \{3, 4, 5\}$$

c) Let B be the event that book 5 is the one selected. What outcomes are in B and find $P(B)$? (5pt)

$B = \{5, 15, 25, 125, 215\}$ and

$$P(B) = \frac{1}{5} + \frac{1}{5}\frac{1}{4} + \frac{1}{5}\frac{1}{4} + \frac{1}{5}\frac{1}{4} + \frac{1}{5}\frac{1}{4}\frac{1}{3} + \frac{1}{5}\frac{1}{4}\frac{1}{3} = \frac{1}{3}$$

d) Let C be the event that book 1 is not examined. What outcomes are in C and find $P(C)$? (10pt)

$C = \{3, 4, 5, 23, 24, 25\}$ and

$$P(C) = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5}\frac{1}{4} + \frac{1}{5}\frac{1}{4} + \frac{1}{5}\frac{1}{4} = \frac{3}{4}$$

[2] A box in a certain supply room contains four 40-W lightbulbs, five 60-W bulbs, and six 75-W bulbs. Suppose that three bulbs are randomly selected.

a) What is the probability that exactly two of the selected bulbs are rated 75 W? (5pt)

$$\frac{C_2^6 C_1^9}{C_3^{15}}$$

b) What is the probability that all three of the selected bulbs have the same rating? (5pt)

$$\frac{C_3^4 C_0^{11}}{C_3^{15}} + \frac{C_3^5 C_0^{10}}{C_3^{15}} + \frac{C_3^6 C_0^9}{C_3^{15}}$$

c) What is the probability that one bulb of each type is selected? (5pt)

$$\frac{C_1^4 C_1^5 C_1^6}{C_3^{15}}$$

d) Suppose now that bulbs are to be selected one by one until a 75-W bulb is found. What is the probability that it is necessary to examine at least six bulbs? (10pt)

Let X denote the round index that the first 75-W is selected. Then $P(X \geq 6) = ?$.

$$P(X \geq 6) = 1 - [P(1) + \ldots + P(5)]$$

$$= 1 - [\frac{6}{15} + \frac{9}{15}\frac{6}{14} + \frac{9}{15}\frac{8}{14}\frac{6}{13} + \frac{9}{15}\frac{8}{14}\frac{7}{13}\frac{6}{12} + \frac{9}{15}\frac{8}{14}\frac{7}{13}\frac{6}{12}\frac{6}{11}]$$

[3] Fifteen telephones have just been received at an authorized service center. Five of these telephones are cellular (CL), five are cordless (CS), and the other five are corded (CD) phones. Suppose that these components are randomly allocated the numbers 1, 2, . . ., 15 to establish the order in which they will be serviced.

a) In how many different ways the numbers can be allocated? (5pt)

$$15!$$

b) What is the probability that all the cordless phones are among the first ten to be serviced? (5pt)

$$\frac{(C_5^{10} 5!)(10!)}{15!}$$

c) What is the probability that after servicing ten of these phones, phones of only two of the three types remain to be serviced? (10pt)

For instance, let us find the probability that the last five phone being serviced were composed of 1 CS and 4 CD, so all five CL's should have been among the first five being serviced, that is,

$$E_1 : \quad 5\,CL + 4\,CS + 1\,CD \quad || \quad 1\,CS + 4\,CD,$$

and the probability of this is

$$P(E_1) = \frac{\left(C_5^{10} 5!5!\right)(5!)\, C_1^5 C_4^5}{15!}$$

Note that the red part is for the fact that we can choose 1 CS and 4 CD's in $C_4^5 C_1^5$ different ways.

Similarly, we can find the probability that the last five includes 2 CS + 3 CD, that is,

$$E_2 : \quad 5\,CL + 3\,CS + 2\,CD \quad || \quad 2\,CS + 3\,CD,$$

and the probability of this:

$$P(E_2) = \frac{\left(C_5^{10}5!5!\right)(5!)\,C_2^5C_3^5}{15!}$$

There are two more cases: last five includes 3 CS + 2 CD or last five includes 4 CS + 1 CD, but we don't actually need to calculate these probabilities as they are completely symmetric to the cases above, so the total probability that the first ten includes all five CL's and the last five includes at least one from both CS and CD is $2(P(E_1) + P(E_2))$, and finally instead of all five CL being in the first ten, all five CS or all five CD could have been in the first five, therefore the answer is

$$6 \cdot (P(E_1) + P(E_2)) \approx 0.25$$

d) What is the probability that two phones of each type are among the first six serviced? (5pt)

Consider one particular pair from each three types for the first six to be serviced, we can order them in 6! different ways, and the remaining ones in 9! different way, so in total, there are 6!9! possible ordering for these three particular pairs. As we can choose one pair from each type in $C_2^5C_2^5C_2^5$ way, the answer is

$$\left(\frac{6!9!}{15!}\right)\left(C_2^5C_2^5C_2^5\right) \approx 0.02$$

[4] Suppose a single gene controls the color of hamsters: black (B) is dominant and brown (b) is recessive. Hence, a hamster will be black unless its genotype is bb.

(Part I) Two hamsters, each with genotype Bb, mate and produce a single offspring. The laws of genetic recombination state that each parent is equally likely to donate either of its two alleles (B or b), so the offspring is equally likely to be any of BB, Bb, bB, or bb (the middle two are genetically equivalent).

a) What is the probability that their offspring has black fur? (5pt)

Note: In this question, Bb and bB are exactly the same things, so the main solution is written under that assumption. However, if you solved the problem under the assumption Bb≠bB, and answer is correct given that assumption, it gets full credit as well.

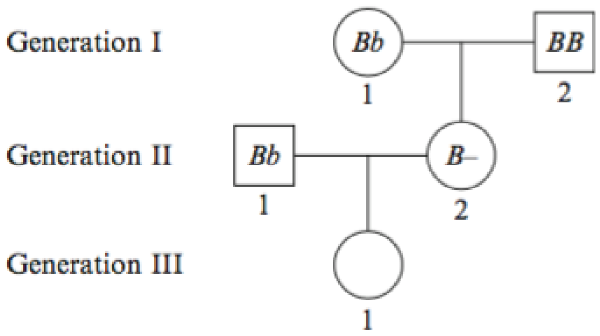
Below, solutions in black ink is for the case Bb=bB, and in red ink for the case Bb≠bB.

$$P(\text{BB or Bb or bB}) = \frac{3}{4}, \quad \frac{3}{4}$$

b) Given that their offspring has black fur, what is the probability its genotype is Bb? (5pt)

$$P\,(\text{Bb}\ \mid (\text{BB or Bb or bB})) = \frac{2}{3}, \quad \frac{1}{3}$$

(Part II) In the figure below, the genotypes of both members of Generation I are known, as is the genotype of the male member of Generation II. We know that hamster II-2 must be black-colored thanks to her father, but suppose that we don't know her genotype exactly (as indicated by B- in the figure).



c) What are the possible genotypes of hamster II-2, and what are the corresponding probabilities? (5pt)

$$\{BB, Bb\}; \quad P(BB) = 1/2, \quad P(Bb) = 1/2$$

$$\{BB, Bb, bB\}; \quad P(BB) = 1/3, \quad P(Bb) = 1/3, \quad P(bB) = 1/3$$

d) If we observe that hamster III-1 has a black coat (and hence at least one B gene), what is the probability that her genotype is Bb? If we later discover (through DNA testing on poor little hamster III-1) that her genotype is BB, what is the probability that her mom is also BB? (10pt)

Case 1: BB

	B	B
B	BB	BB
b	bB	bB

Case 2: Bb

	B	b
B	BB	Bb
b	bB	bb

Case 3: bB

	b	B
B	Bb	BB
b	bb	bB

$$P(\text{her genotype is Bb} \mid \text{ Black Coat}) = \frac{4}{7}, \quad \frac{2}{10}$$

and

$$P(\text{her mom is BB } \mid \text{ her genotype is BB}) = \frac{2}{3}, \quad \frac{2}{4}$$