Problem 10 - Solutions

Statistics - NYU, Summer 2016
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[1] a) $E(R_p) = (0.5)(0.1) + (0.5)(0.2) = 0.15$
$Var(R_p) = (0.5)^2(0.2)^2 + (0.5)^2(0.4)^2 + 2(0.5)(0.5)(0.03) = 0.065$

b) R_p is normally distributed and $R_p \sim N(0.15, 0.065)$.

c)

$$\begin{aligned} P(R_p < 0.329) &= P(\frac{R_p - 0.15}{\sqrt{0.065}} < \frac{0.329 - 0.15}{\sqrt{0.065}}) \\ &= P(z < 0.7) \\ &= 0.758 \end{aligned}$$

So $P(R_p > 0.329) = 1 - 0.758 = 0.242$

d) Yes, because new portfolio has the same mean return but smaller variance (or uncertainty).

[2] a) $SAT \sim N(1518, 325^2)$ and $ACT \sim N(21.1, 4.8^2)$
$P(SAT_{67} > SAT) = 1 - 0.67 = 0.33$ or equivalently expressed in terms of Z socores $\frac{SAT_{67} - 1518}{325} = 0.44$ which yields $SAT_{67th} = 1661$.

b) Since these two tests measure the same aptitude this person should get again the 67th percentile. Similar algebra: $\frac{ACT_{67} - 21.1}{4.8} = 0.44$ which yields $ACT_{67th} = 23.2$

c) Z-scores of these two tests should be the same since they should correspond to the same percentile. Therefore,

$$\frac{1900 - 1518}{325} = \frac{ACT - 21.1}{4.8} \implies ACT = 26.74$$

d) Let $\bar{X}$ be sample mean of 25 randomly selected ACT scores. Then the sampling distribution will be $\bar{X} \sim N(21.1, (\frac{4.8}{5})^2)$ and

$$P(\bar{X} > 20) = P(Z > \frac{20 - 21.1}{4.8/5}) = P(Z > -1.145) = 0.8749$$

[3] a) Since manufacturer only concerns whether the impurity concentration in pills exceeds 3%, one-sided alternative test is more appropriate.

b) $H_0 : \mu \leq 3$
$H_1 : \mu > 3$
Remark: Although, in general, we put the claim that we want to support in H_1, here it is not appropriate. Because if we use $H_1 : \mu < 3$, then we should have $H_0 : \mu \geq 3$. But as $\bar{X} = 3.07$ is already greater than 3 we can not reject the null hypothesis, and therefore this test would be degenerate (i.e. does not test anything in fact). Hypothesis testing can be informative only if H_0 is falsifiable.

Since $t_{63} = \frac{3.07-3.0}{0.45/\sqrt{64}} = 1.244 < t_{63,0.05} = 1.671$ we fail to reject H_0 at the 5% level.

(Note that instead of $t_{63,0.05}$, which is not present in the t-table, we used $t_{60,0.05} = 1.671$ as a close approximation.)

c) Since the degrees of freedom is high, standard normal distribution gives a good approximation for t-distribution, and therefore we can use z-table to calculate the p-value.
$p - value = 1 - F(1.244) = 1 - .8925 \approx 0.1075$

d) **Step 1:** Find $\overline{X}_c$ that we would fail to reject H_0.
From part (b) this is given by the equation

$$\frac{\overline{X}_c - 3.0}{0.45/\sqrt{64}} = 1.671$$

which gives $\overline{X}_c = 3.09$. So for any $\overline{X} \leq 3.09$ we fail to reject H_0.

Step 2: Under the true population distribution assumption, find the probability that $\overline{X} \leq \overline{X}_c$.
We need to find $\overline{X} \leq 3.09$, or equivalently

$$P(t_{63} < \frac{3.09 - 3.2}{0.45/\sqrt{64}}) = P(t_{63} < -1.88) = ?$$

Note that ideally we have to use t-distribution, but t-table is not detailed enough to find this probability. From t-table we can only say that this probability is between 0.01 and 0.05 (make sure that you understand why, go to t-table to see this is true when degrees of freedom is 60). However, since n is large we can use Z-table to find that probability. Namely, we now think of the problem as finding: $P(Z < -1.88) =?$. From the Z-table we find that $P(Z < -1.88) = 0.03$. This is our type-II error (probability of failing to reject a false hypothesis, which we denote by β), i.e., $\beta = 0.03$. Finally, the power of our test is $1 - \beta = 1 - 0.03 = .97$.

Just to clarify, we use t-distribution when we don't know the population variance (σ^2). But when n is large enough, above 60 for example, we might use Z-distribution instead. For example, suppose you need to find $P(t_{74} <$

1.5). Here, $n = 75$ is large so we can use Z-distribution instead and can use $P(Z < 1.5)$.

e) The p-value would be higher-the graph should show that the p-value now corresponds to the area in both of the tails of the distribution whereas before it was the area in one of the tails.

[4] a) Since $t_{15} = \frac{\overline{X}-10.1}{s/\sqrt{n}} = \frac{10.4-10.1}{0.4/\sqrt{16}} = 3 > t_{15,0.05} = 1.753$, reject the null hypothesis in favor of the alternative hypothesis.

Interpretation: with 95% confidence, we conclude that the mean weight of cereal in the population is greater than 10.1 ounce.

b) Since $t_{15,0.005} = 2.947$ and $t_{15,0.001} = 3.733$, the $p-value$ should be something very close to 0.5%. Note that in part a we had rejected H_0 with 95% confidence level, but $p - value$ shows that in fact we can reject H_0 with almost 99.5% confidence!

[5] The hypothesis test assumes that the population is normally distributed
$H_0 : \sigma \leq 2$
$H_1 : \sigma > 2$
Decision rule: Reject H_0 if $\chi^2 = \frac{(n-1)s^2}{\sigma^2} > \chi^2_{19,0.05}$. Since $\chi^2 = \frac{(19)(2.36)^2}{(2)^2} = 26.45 < \chi^2_{19,0.05} = 30.14$, fail to reject H_0 at the 5% level.

Remark: Although the sample standard deviation is greater than 2, this does NOT allow us to say that population variance is greater than 2 with 95% confidence. Probably we could say the population variance is greater than 2 if we were willing to work with a lower confidence level, and in fact we could determine this level exactly by using the p-value approach.

[6] a) $H_0 : \mu \geq 29$
$H_1 : \mu < 29$

b) Test statistics: $t_8 = \frac{(26-29)}{(6.2/3)} = -1.45$
Since $-1.45 > -t_{8,0.05} = -1.860$, we are unable to reject at $\alpha = 0.05$. There is no sufficient evidence to disprove claim that the product will increase output per machine by 29 units per hour.

c) Our test statistics, $t-$value in part (c), is -1.45. If we look at the t-table for 8 degrees of freedom, we see that $-1.860 = t_{8,0.05} < -1.45 < -1.397 = t_{8,0.1}$, therefore $p - value$ is greater than 5% and less than 10%, which means we would be able to reject the null-hypothesis in the previous part only for $\alpha = 0.10$.

d) $H_0 : \mu \leq 29, \quad H_1 : \mu > 29$
So that we can state whether the manager's claim is valid with certain level of confidence.

[7] a) $P(s > 0.8\sigma) = P(\frac{s^2}{\sigma^2} > 0.64) = P(\frac{(n-1)s^2}{\sigma^2} > 0.64(n-1))$
$= P(\chi^2_{23} > (0.64)(23)) = P(\chi^2_{23} > 14.72) \approx 0.90$

b) $P(s^2 < k\sigma^2) = P(\frac{(n-1)s^2}{\sigma^2} < 23k) = 0.90$
$\Rightarrow P(\chi^2_{23} < 23k) = 0.90 \Rightarrow \quad 23k = 32.01 \quad \Rightarrow k = 1.391$

c) $H_0 : \sigma^2 \leq 15^2$
$H_1 : \sigma^2 > 15^2$
Test statistics: $\chi^2 = \frac{23.20^2}{15^2} = 40.89$
Critical value: $\chi^2_{23,.05} = 35.17$
$\Rightarrow$ Reject H_0: There is enough evidence that population standard deviation is greater than 15.

[8] a) $\overline{X} = 2.1$ is the best point estimate.

b) $\overline{X} \pm t_{40,0.05}\frac{s}{\sqrt{n}} = 2.1 \pm (1.684)\frac{4.8}{\sqrt{41}}$
$\Longrightarrow$ 90% C.I. for μ : $\quad 0.84 < \mu < 3.36$

c) State the null and the alternative hypothesis:
$H_0 : \mu = 3.2$
$H_1 : \mu \neq 3.2$
Calculate the test-statistics:
$t_{40} = \frac{\overline{X}-\mu}{s/\sqrt{n}} = \frac{2.1-3.2}{4.8/\sqrt{41}} = -1.47$
Find the critical table value:
$-t_{40,0.05} = -1.684$
Conclude:
as $-1.684 < -1.47$, we fail to reject the null hypothesis that the mean weight loss is 3.2.

d) Yes, because the test in part (c) is two-sided and the claimed μ lies in the C.I. constructed in part (b). The minimum mean weight loss that would be rejected in part (c) is 3.36.

e) We can decide the effectiveness of the test by concluding the following hypothesis testing:
$H_0 : \mu \geq 3.2$
$H_1 : \mu < 3.2$
i.e. if we fail to reject H_0, then the program is effective.
Test-statistics is the same:
$t_{40} = \frac{\overline{X}-\mu}{s/\sqrt{n}} = \frac{2.1-3.2}{4.8/\sqrt{41}} = -1.47$
but the critical table value now is $-t_{40,0.10} = -1.303$.
Since $-1.47 < -1.303$, we reject the null hypothesis that the mean weight loss is at least 3.2 lb, and therefore we can not conclude that the program is effective.

f) Since the hypothesis of interest here is one-sided we could not use the C.I. constructed in part (b).

[9] a) $H_0 : \mu \geq 21.1$
$H_1 : \mu < 21.1$

b) Since $t_{24} = \frac{13.2-21.1}{3.7/5} = -10.68 < -t_{24,0.05} = -1.711$ reject H_0. There is sufficient evidence that filtered cigarettes have a mean tar amount less than 21.1 mg. The results suggest that the filters are effective in reducing the amount of tar.

c) We can not find an exact p-value from the t table, but from $t = -10.68$ we can say that $p - value < 0.001$.

d) $t_{24} = \frac{\overline{X}_c - 21.1}{3.7/\sqrt{25}} = -1.711 \Longrightarrow \overline{X}_c = 19.83$ and then
$\beta = P(t_{24} > \frac{19.83-17.98}{3.7/\sqrt{25}}) = P(t_{24} > 2.5) \approx 0.01$.
Therefore the power of the test is $1 - \beta = 99\%$

e) $H_0 : \sigma^2 = 8$
$H_1 : \sigma^2 \neq 8$
$\Longrightarrow$ 90% C.I. for σ^2: $\frac{(24)(3.7)^2}{\chi^2_{24,0.05}} < \sigma^2 < \frac{(24)(3.7)^2}{\chi^2_{24,0.95}}$
And since the C.I. interval $9 < \sigma^2 < 23.7$ doesn't contain 8, we reject H_0 that $\sigma^2 = 8$.

[10] a) $H_0 : \mu \geq 20,000$
$H_1 : \mu < 20,000$

b) Test statistics $t_{29} = \frac{19{,}597-20{,}000}{1{,}103/\sqrt{30}} = -2.001$, and the critical value is $t_{29,0.05} = 1.699$.
Since $-2.001 < -1.699$, we reject the null hypothesis.

c) $p - value = P(t_{29} < -2.001)$, which implies p-value is between .05 and .025. Therefore, reject the null for $\alpha = .05$ or .10 but fail to reject for $\alpha = .01$.

d) Step 1:
$\frac{\overline{X}_c - 20{,}000}{1{,}103/\sqrt{30}} = -t_{29,0.05} = -1.699 \Rightarrow \overline{X}_c = 19,657.9$
Step 2:
$\beta = P(\overline{X} > \overline{X}_c | \mu = 19,310) = P(t_{29} > \frac{19{,}657.9-19{,}310}{1{,}103/\sqrt{30}}) = P(t_{29} > 1.727) \approx 0.05$ Power $= 1 - \beta = \approx 95\%$

e) 95 % C.I. for μ:
$\overline{X} \mp t_{29,0.025} \frac{1{,}103}{\sqrt{30}} = 19,597 \mp 2.045 \frac{1{,}103}{\sqrt{30}} = [19,186 \quad 20,008]$

f) The confidence interval is two-sided, it can not be used to conclude a two-sided test as is the case here. Otherwise, note that we would reach a result contradictory to the part (b).