Problem Set 1- Solutions

Applied Statistics and Econometrics II
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Ercan Karadas

[1] Let $Y_n = X_n/n$. Then

$$E(Y_n) = \frac{1}{n}(np) = p = \mu_Y$$
$$Var(Y_n) = \frac{1}{n^2}np(1-p) = \frac{1}{n}p(1-p) = \sigma_Y^2$$

We want to show for a given $\varepsilon > 0$

$$P\left(|Y_n - p| \geq \varepsilon\right) \to 0 \qquad \text{as } n \to \infty$$

a) Chebyshev's inequality

$$P\left(|Y_n - \mu_Y| \geq k\sigma_Y\right) \leq \frac{1}{k^2}$$
$$\implies P\left(|Y_n - p| \geq \varepsilon\right) \leq \frac{\sigma_Y^2}{\varepsilon^2} = \frac{1}{\varepsilon^2}\frac{1}{n}p(1-p) \qquad (k\sigma_Y = \varepsilon \implies k = \varepsilon/\sigma_Y)$$

As $n \to \infty$, RHS $\to 0$, therefore

$$P\left(|Y_n - p| \geq \varepsilon\right) \to 0$$

as desired.

b) Note that we can write

$$X_n = \sum_{i=1}^{n} X_i$$

where X_i is a Bernouilli random variable with parameter p.
According to WLL

$$\frac{1}{n}X_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} E(X_i) = p$$

[2] a) θ is the lower bound of the set of all possible values, so as the sample size increases one would expect the minimum of the sample (Y_n) gets closer to the parameter of interest.

b) For $t < \theta$, $F_{Y_n}(t) = 0$ and for $t \geq \theta$

$$\begin{aligned} F_{Y_n}(t) &= P(Y_n \leq t) \\ &= 1 - P(\min\{X_1, \ldots, X_n\} > t) \\ &= \prod_{i=1}^{n} P(X_i > t) \\ &= [P(X_i \leq t)]^n \qquad \text{(Random variables are i.i.d)} \\ &= 1 - \left[\int_\theta^t e^{-(s-\theta)} ds\right]^n \\ &= 1 - e^{-(t-\theta)n} \end{aligned}$$

Therefore,

$$F_{Y_n}(t) = \begin{cases} 0 & t < \theta \\ 1 - e^{-(t-\theta)n} & \theta \leq t \end{cases}$$

c)

$$Y_n \xrightarrow{p} \theta \iff \lim_{n\to\infty} P(|Y_n - \theta| \leq \varepsilon) = 1$$

$$\begin{aligned} P(|Y_n - \theta| \leq \varepsilon) &= P(\theta - \varepsilon \leq Y_n \leq \theta + \varepsilon) \\ &= F(\theta + \varepsilon) - F(\theta - \varepsilon) \\ &= (1 - e^{-\varepsilon n}) - 0 \\ &= 1 - e^{-\varepsilon n} \to 1 \qquad \text{as } n \to \infty \end{aligned}$$

d) $f_{Y_n}(t) = \frac{\partial F_{Y_n}(t)}{\partial t} = ne^{-(t-\theta)n}$

$$\begin{aligned} E(Y_n) &= \int_\theta^\infty t f_{Y_n}(t) dt \\ &= \int_\theta^\infty tne^{-(t-\theta)n} dt \\ &= \theta + \frac{1}{n} \neq \theta \end{aligned}$$

Hence, Y_n is a biased estimator of θ.

e) An unbiased estimator of θ: $Z_n = Y_n - \frac{1}{n}$.

[3] a)

$$\begin{aligned} S_n^2 &= \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 \\ &= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}_n^2 \end{aligned}$$

By WLL $\bar{X}_n \xrightarrow{p} \mu$, and therefore we have $\bar{X}_n^2 \xrightarrow{p} \mu^2$. On the other hand,

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{p} E(X_i^2) = \sigma^2 + \mu^2$$

Combining these two results gives

$$S_n^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}_n^2 \xrightarrow{p} \sigma^2 + \mu^2 - \mu^2 = \sigma^2 \, .$$

b) From part (a) it's immediate that

$$\sqrt{n}\left(S_n^2 - \sigma^2\right) = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}_n^2 - \sigma^2\right)$$

c) Since the distribution of s^2 doesn't depend on μ, wlog assume that $\mu = 0$. Then by the CLT

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \sigma^2\right) \xrightarrow{d} N(0, \nu^2)$$

On the other hand, $\bar{X}_n^2 \xrightarrow{p} 0$ and therefore $\sqrt{n}\bar{X}_n^2 \xrightarrow{p} 0$. Combining these two gives

$$\sqrt{n}\left(S_n^2 - \sigma^2\right) = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}_n^2 - \sigma^2\right) \xrightarrow{d} N(0, \nu^2)$$

[4] a) Since $X_i \sim \text{Ber}(p)$, $E(X_i) = p$ and $Var(X_i) = p(1-p)$, it follows from the CLT that

$$\sqrt{n}\left(\bar{X}_n - p\right) \xrightarrow{d} N\left(0, p(1-p)\right)$$

b) Suppose $Y_i \sim \text{Ber}(p)$, then from part (a)

$$\sqrt{n}\left(\bar{Y}_n - p\right) \xrightarrow{d} N\left(0, p(1-p)\right)$$

But since $X_n \sim \text{Bin}(n, p)$, we can write $X_n = \sum_{i=1}^{n} Y_i$, where $Y_i \sim \text{Ber}(p)$. Now, noting that

$$\bar{Y}_n = \frac{\sum_{i=1}^{n} Y_i}{n} = \frac{X_n}{n}$$

gives the result

$$\sqrt{n}\left(\frac{X_n}{n} - p\right) = \sqrt{n}\left(\bar{Y}_n - p\right) \xrightarrow{d} N\left(0, p(1-p)\right)$$

c) Since $X_n \sim \text{Bin}(k_n, p)$, from part (b)

$$\sqrt{k_n}\left(\frac{X_n}{k_n} - p\right) \xrightarrow{d} N\left(0, p(1-p)\right)$$

$$\implies \frac{X_n}{k_n} \xrightarrow{d} N\left(p, \frac{p(1-p)}{k_n}\right)$$

$$\implies X_n \xrightarrow{d} N\left(k_n p, k_n p(1-p)\right)$$

[5] a) Since $X \sim \text{Bin}(n, p^2)$, from 4.b we have the result

$$\sqrt{n}\left(\frac{X}{n} - p^2\right) \xrightarrow{d} N\left(0, p^2(1-p^2)\right)$$

b) Since $Y \sim \text{Bin}(n, p)$, from 4.b we have

$$\sqrt{n}\left(\frac{Y}{n} - p\right) \xrightarrow{d} N\left(0, p(1-p)\right)$$

Define $f(x) = x^2$, then by the delta method

$$\sqrt{n}\left(f\left(\frac{Y}{n}\right) - f(p)\right) \xrightarrow{d} N\left(0, p(1-p) \cdot [f'(p)]^2\right)$$

Substituting $f'(p) = 4p^2$ gives the result.

c) Comparing the variances of the estimators in two cases tells which estimator is better in terms of having a smaller variance:

$$p^2(1-p^2) < p(1-p)4p^2$$

solving this inequality we conclude if $p > 1/3$, X/n is preferable and if $p < 1/3$, $(Y/n)^2$ is preferable.

[6] a) Note that we can write $X_n = \sum_{i=1}^{n} Y_i$, where $Y_i \sim \text{Poisson}(\lambda)$. Therefore, $E(Y_i) = \lambda$ and $Var(Y_i) = \lambda$, and by the CLT

$$\sqrt{n}\left(\frac{X_n}{n} - \lambda\right) = \sqrt{n}\left(\frac{\sum_{i=1}^{n} Y_i}{n} - \lambda\right) \xrightarrow{d} N(0, \lambda)$$

b) Define $f(x) = \sqrt{x}$, then $f'(\lambda) = 1/2\sqrt{\lambda}$. Using part (a) and the delta method gives

$$\sqrt{n}\left(\sqrt{\frac{X_n}{n}} - \sqrt{\lambda}\right) \xrightarrow{d} N\left(0, \lambda\left[\frac{1}{2\sqrt{\lambda}}\right]^2\right) = N\left(0, \frac{1}{4}\right)$$

[7] a) A second-order Taylor expansion of g about θ:

$$g(X_n) = g(\theta) + g'(\theta)\left(X_n - \theta\right) + \frac{1}{2}g''(\theta)\left(X_n - \theta\right)^2 + \text{Remainder Term}$$

using $g'(\theta) = 0$ and then rearranging the terms gives

$$g(X_n) - g(\theta) = \frac{1}{2}g''(\theta)\left(X_n - \theta\right)^2 + \text{Remainder Term}$$

b) For convenience let's drop the remainder term and then multiply both sides by n to get

$$n\left[g(X_n) - g(\theta)\right] = \frac{1}{2}g''(\theta)\left[\sqrt{n}\left(X_n - \theta\right)\right]^2$$

Note that the main hypothesis can be expressed as

$$\frac{1}{\sigma}\sqrt{n}\,(X_n-\theta) \xrightarrow{d} N(0,1) \equiv Z$$

Then by Slutsky theorem

$$\left[\frac{1}{\sigma}\sqrt{n}\,(X_n-\theta)\right]^2 \xrightarrow{d} Z^2 = \chi_1^2$$

$$\Longrightarrow \left[\sqrt{n}\,(X_n-\theta)\right]^2 \xrightarrow{d} \sigma^2\chi_1^2$$

Therefore we have

$$n\left[g(X_n)-g(\theta)\right] = \frac{1}{2}g''(\theta)\left[\sqrt{n}\,(X_n-\theta)\right]^2 \xrightarrow{d} \frac{1}{2}\sigma^2 g''(\theta)\chi_1^2\,.$$

[8] First note that we have this result by the CLT

$$\sqrt{n}\left[\frac{X_n}{n}-p\right] \xrightarrow{d} N(0,p(1-p))$$

Now, in order to be able to apply the delta method we must choose

$$c = g(p) = \left(\frac{1}{\sqrt{3}}\right)^3 - \frac{1}{\sqrt{3}} = -\frac{2\sqrt{3}}{9}$$

Also compute that $g'(\frac{1}{\sqrt{3}}) = 0$ and $g''(\frac{1}{\sqrt{3}}) = 2\sqrt{3} \neq 0$. Now note that the conditions of the previous exercise are satisfied so we can apply that result to write

$$n\left[g\left(\frac{X_n}{n}\right)-c\right] \xrightarrow{d} \frac{1}{2}\sigma^2 g''(p)\chi_1^2\,.$$

Finally, substituting $c = -\frac{2\sqrt{3}}{9}$, $g''(\frac{1}{\sqrt{3}}) = 2\sqrt{3}$, and $\sigma^2 = p(1-p) = \frac{1}{\sqrt{3}}\left(1-\frac{1}{\sqrt{3}}\right)$ we obtain

$$n\left[g\left(\frac{X_n}{n}\right)+\frac{2\sqrt{3}}{9}\right] \xrightarrow{d} \left(1-\frac{1}{\sqrt{3}}\right)\chi_1^2\,.$$