Problem Set 7

Applied Statistics and Econometrics II
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[1] Consider the time series

$$x_t = \beta_1 + \beta_2 t + w_t$$

where β_1 and β_2 are known constants and w_t is a white noise process with variance σ_w^2.

a) Determine whether x_t is stationary.

b) Show that the process $y_t = x_t - x_{t-1}$ is stationary.

c) Show that the mean of the moving average

$$v_t = \frac{1}{2q+1} \sum_{j=-q}^{q} x_{t-j}$$

is $\beta_1 + \beta_2 t$, and give a simplified expression for the autocovariance function.

[2] For a moving average process of the form

$$x_t = w_{t-1} + 2w_t + w_{t+1}$$

where w_t are are independent with zero means and variance σ_w^2, determine the autocovariance and autocorrelation functions as a function of lag $h = s - t$ and plot the ACF as a function of h.

[3] Consider the random walk with drift model

$$x_t = \delta + x_{t-1} + w_t$$

for $t = 1, 2, \ldots$ with $x_0 = 0$, where w_t is white noise with variance σ_w^2.

a) Show that the model can be written as $x_t = \delta t + \sum_{k=1}^{t} w_k$.

b) Find the mean function and the autocovariance function of x_t.

c) Argue that x_t is not stationary.

d) Show $\rho_x(t-1, t) = \sqrt{\frac{t-1}{t}}$ as $t \to \infty$. What is the implication of this result.

e) Suggest a transformation to make the series stationary, and prove that the transformed series is stationary.

[4] Let x_t be a stationary normal process with mean μ_x and autocovariance function $\gamma(h)$. Define the nonlinear time series

$$y_t = exp\{x_t\}\,.$$

a) Express the mean function $E(y_t)$ in terms of μ_x and $\gamma(0)$. *Hint:*The moment generating function of a normal random variable x with mean μ and variance σ^2 is

$$M_x(\lambda) = E[exp\{\lambda x\}] = exp\left\{\mu\lambda + \frac{1}{2}\sigma^2\lambda^2\right\}\,.$$

b) Determine the autocovariance function of y_t . The sum of the two normal random variables $x_{t+h} + x_t$ is still a normal random variable.

[5] a) Simulate a series of $n = 500$ Gaussian white noise observations as in Example 1.8 and compute the sample ACF, $\hat{\rho}(h)$, to lag 20. Compare the sample ACF you obtain to the actual ACF, $\rho(h)$. [Recall Example 1.19.]

b) Repeat part (a) using only $n = 50$. How does changing n affect the results?

c) Simulate a series of $n = 500$ moving average observations as in Example 1.9 and compute the sample ACF, $\hat{\rho}(h)$, to lag 20. Compare the sample ACF you obtain to the actual ACF, $\rho(h)$. [Recall Example 1.20.]

d) Repeat part (a) using only $n = 50$. How does changing n affect the results?

[6] Consider a process consisting of a linear trend with an additive noise term con- sisting of independent random variables w_t with zero means and variances σ_w^2, that is

$$x_t = \beta_1 + \beta_2 t + w_t$$

where β_1 and β_2 are fixed constants.

a) Prove x_t is stationary.

b) Prove that the first difference series $\Delta x_t = x_t - x_{t-1}$ is stationary by finding its mean and autocovariance function.

c) Repeat part (b) if w_t is replaced by a general stationary process, say y_t, with mean function μ_y and autocovariance function $\gamma_y(h)$.

[7] Let $\{w_t; t = 0, 1, \ldots\}$ be a white noise process with variance σ_w^2 and let $|\phi| < 1$ be a constant. Consider the process $x_0 = w_0$, and

$$x_t = \phi x_{t-1} + w_t, \qquad t = 1, 2, \ldots.$$

We might use this method to simulate an AR(1) process from simulated white noise.

a) Show that $x_t = \sum_{j=0}^{t} \phi^j w_{t-j}$ for any $t = 0, 1, \ldots$.

b) Find the $E(x_t)$.

c) Show that for $t = 0, 1, \dots,$

$$var(x_t) = \frac{\sigma_w^2}{1-\phi^2}\left(1 - \phi^{2(t+1)}\right)$$

d) Show that, for $h \geq 0$,

$$cov(x_{t+h}, x_t) = \phi^h var(x_t)$$

e) Is x_t stationary?

f) Argue that, as $t \to \infty$, the process becomes stationary, so in a sense, x_t is "asymptotically stationary."

g) Comment on how you could use these results to simulate n observations of a stationary Gaussian AR(1) model from simulated iid $N(0, 1)$ values.

h) Now suppose $x_0 = w_0/\sqrt{1-\phi^2}$. Is this process stationary? *Hint:* Show $var(x_t)$ is constant.

[8] Identify the following models as $ARMA(p, q)$ models (watch out for parameter redundancy), and determine whether they are causal and/or invertible:

a) $x_t = 0.80x_{t-1} - 0.15x_{t-2} + w_t - 0.30w_{t-1}$.

b) $x_t = x_{t-1} - 0.50x_{t-2} + w_t - w_{t-1}$.

[9] Consider the $ARMA(1, 1)$ process $x_t = \phi x_{t-1} + \theta w_{t-1} + w_t$, where $|\phi| < 1$.

a) Show that the ACF is

$$\rho(h) = \frac{(1+\theta\phi)(\phi+\theta)}{1+2\theta\phi+\theta^2}\phi^h, \qquad h \geq 1\,.$$

b) Compare the form with that of the ACF for the $ARMA(1, 0)$ and the $ARMA(0, 1)$ series. Plot the ACFs of the three series on the same graph for $\phi = .6, \theta = .9$, and comment on the diagnostic capabilities of the ACF in this case.

c) Generate $n = 100$ observations from each of the three models discussed in the previous problem. Compute the sample ACF for each model and compare it to the theoretical values. Compute the sample PACF for each of the generated series and compare the sample ACFs and PACFs with the general results given in Table 3.1 in the book.

[10] Consider the following model with lagged endogenous variables and correlated error term,

$$y_t = [\mathbf{z}_t' \quad y_{t-1}]\boldsymbol{\beta} + u_t, \qquad u_t = \alpha u_{t-1} + v_t$$

where v_ts are IDD, zero mean and $E[v_t z_j] = 0$ for all t, j and set $\beta = \left(\boldsymbol{\beta}_z', \beta_y\right)'$.

a) Under which conditions u_t strictly and covariance stationary? Find the autocovariance function of u_t in the latter case and calculate

$$\lim_{T\to\infty} \text{Var}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} u_t\right].$$

b) Study the consistency properties of the OLS estimate of β.

c) Define a GMM estimate of β exploiting the restrictions $E[v_t z_j] = 0$ for all t, j, and study its consistency and asymptotic distribution.

d) Study the consistency and asymptotic distribution of the OLS estimate of β, using the transformed model

$$y_t^* = [\mathbf{z}_t^{*'} \quad y_{t-1}^*]\boldsymbol{\beta} + u_t^*, \qquad t = 2, 3, \ldots, T,$$

where

$$\begin{aligned} y_t^* &= y_t - \alpha y_{t-1} \\ \mathbf{z}_t^* &= \mathbf{z}_t - \alpha \mathbf{z}_{t-1} \\ u_t^* &= u_t - \alpha u_{t-1}\,. \end{aligned}$$