Problem Set 8

Applied Statistics and Econometrics II
Spring 2018, NYU
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(Sent by 3pm, April 25)

[1] Consider the following data generating process

$$\begin{aligned} x_t + \beta y_t &= u_{1t}, &\quad \text{where} \quad u_{1t} &= \theta u_{1,t-1} + \epsilon_{1t} \\ x_t + \alpha y_t &= u_{2t}, &\quad \text{where} \quad u_{2t} &= \rho u_{2,t-1} + \epsilon_{2t} \end{aligned}$$

where $|\rho| < 1$ and

$$\begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix} \sim D\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \gamma \\ \gamma & \sigma_1^2 \end{pmatrix}\right]$$

where D denotes a generic distribution.

a) Derive the degree of integratedness of the two series, x_t and y_t, explicitly stating the parameter restrictions required in each case.
b) Under what coefficient restrictions are x and y cointegrated? What are the cointegrating vectors in these cases?
c) Choose a particular set of coefficients that ensures x and y are cointegrated and derive the following representations:
    i. the moving-average: that is $(x_t\, y_t)'$ on the left hand side, $(\epsilon_{1t}\, \epsilon_{2t})'$ and its lags on the right hand side.
    ii. The autoregressive in the levels: that is, $(x_t\, y_t)'$ on the right hand side, and lags of $(x_t\, y_t)'$ and $(\epsilon_{1t}\, \epsilon_{2t})'$ on the right hand side.
    iii. The Error-Correction Representation: that is, $(\Delta x_t\, \Delta y_t)'$ as a function of z_{t-1} and residuals (no need to be specific about the residuals).
d) In general, discuss the pros and cons of obtaining impulse responses from a VAR estimated in the levels, as in part (c)-(i) and one in VEC form, as in part (c)-(iii).

[2] Consider the following data generating process

$$\begin{aligned} x_t + y_t &= v_t, &\quad v_t(1 - \rho_1 L) &= \epsilon_{1t} \\ 2x_t + y_t &= u_t, &\quad u_t(1 - \rho_2 L) &= \epsilon_{2t} \end{aligned}$$

where $|\rho| < 1$ and

$$\begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_1^2 \end{pmatrix}\right]$$

where D denotes a generic distribution.

a) Determine whether this system is stationary, non-stationary, or non-stationary but cointegrated according to the following scenarios:

   i. $|\rho_1| < 1$, $|\rho_2| < 1$

   ii. $|\rho_1| = 1$, $|\rho_2| < 1$

   iii. $|\rho_1| = 1$, $|\rho_2| = 1$

b) Obtain the reduced-form, autoregressive representation of the system in the levels when it is cointegrated.

c) Given the representation that you have just found, calculate the reduced-form, impulse response function coefficient matrices, Ψ_s for periods $s = 0, 1$, and 2 for $\rho_2 = 0.5$. What is Ψ_s as $s \to \infty$? Given the reduced form impulse response matrices, calculate the structural impulse responses for periods 0, 1, 2. What is happening as $s \to \infty$? Explain this result.

d) Find the moving-average representation of the system and the cointegrating vector when it is cointegrated.

e) Describe how you could estimate the cointegrating vector in a regression of y_t on x_t which is well behaved in small samples.

[3] Consider the following bivariate VAR

$$y_{1t} = 0.3y_{1t-1} + 0.8y_{2t-1} + \epsilon_{1t}$$
$$y_{2t} = 0.9y_{1t-1} + 0.4y_{2t-1} + \epsilon_{2t}$$

where $E(\epsilon_{1t}\epsilon_{1\tau}) = 1$ for $t = \tau$ and 0 otherwise; $E(\epsilon_{2t}\epsilon_{2\tau}) = 2$ for $t = \tau$ and 0 otherwise; and $E(\epsilon_{1t}\epsilon_{2\tau}) = 0$ for all t, τ.

Answer the following questions.

a) Is this system covariance-stationary?

b) Calculate $\Psi_s = \frac{\partial y_{t+s}}{\partial \epsilon_t'}$ for $s = 0, 1, 2$. What is the limiting expression as $s \to \infty$.

c) Calculate the fraction of the MSE of the two period-ahead forecast error for variable 1, $E[y_{1,t+2} - \hat{E}(y_{1,t+2}|y_t, y_{t-1}, \ldots)]^2$, that is due to $\epsilon_{1,t+1}$ and $\epsilon_{1,t+2}$.

[4] Consider the following VAR

$$y_t = (1 + \beta)y_{t-1} - \beta\alpha x_{t-1} + \epsilon_{1t}$$
$$x_t = \gamma y_{t-1} + (1 - \gamma\alpha)x_{t-1} + \epsilon_{2t}$$

Answer the following questions.

a) Show that this VAR is not-stationary.

b) Find the cointegrating vector and derive the VECM representation.

c) Transform the model so that it involves the error correction term (call it z) and a difference stationary variable (call it Δw_t). w will be a linear combination of x and y but should not contain z. *Hint:* the weights in this linear combination will be related the coefficients of the error correction terms.

d) Verify that y and x can be expressed as a linear combination of w and z. Give an interpretation as a decomposition of the vector $(y\, x)'$ into permanent and transitory components.

[5] Consider the following VECM

$$\Delta y_t = c + \alpha\beta' y_{t-1} + \epsilon_t, \qquad \epsilon_{it} \sim iid\,(0, \sigma^2)$$

where $\alpha = (\alpha_1, 0)'$ and $\beta = (1, -\beta_2)'$. Equation by equation, the system is given by

$$\begin{aligned}\Delta y_{1t} &= c_1 + \alpha_1(y_{1,t-1} - \beta_2 y_{2,t-1}) + \epsilon_{1t} \\ \Delta y_{2t} &= c_2 + \epsilon_{2t}\end{aligned}$$

Answer the following questions.

a) From the VECM representation above, derive the VECM representation

$$\Delta y_t = c + \Pi y_{t-1} + \epsilon_t$$

and VAR representation

$$y_t = c + A y_{t-1} + \epsilon_t$$

b) Based on the given values of the elements in α and β, determine $\alpha_\perp$ and $\beta_\perp$ such that $\alpha'\alpha_\perp = 0$ and $\beta'\beta_\perp = 0$.

c) Using the Granger representation theorem determine that $\Psi(1) = \beta_\perp \left(\alpha'_\perp I_2 \beta'_\perp\right)^{-1} \alpha_\perp$, where $\Psi(L)$ is the moving average polynomial corresponding to the VECM system above and I_2 is the identity matrix of order 2. *Hint:* you may show this result by showing that $\Psi(1)$ is orthogonal to the cointegrating space.

d) Using the Beveridge-Nelson decomposition and the result in (c), determine the common trend in the VECM system.

e) Show that $\beta' y_t$ follows an AR(1) process and show that this AR(1) is stable provided that $-2 < \alpha_1 < 0$. What can you say about the system when $\alpha_1 = 0$?

[6] Consider the following process,

$$y_t \sim \ln(\beta y_{t-1}) + \epsilon_t, \quad y_0 = 1, \quad \epsilon_t \sim iidD(0, 1), \quad \beta > 0$$

a) Discuss how you would determine whether this process is stationary (although it is not necessary to derive this analytically, it is a bonus if you can determine for what values of β will the process be stationary).

b) Assuming the regularity conditions for QMLE are met, find the conditional QMLE estimator for β (i.e., conditional on $y_0 = 1$). Assume the variance of the ϵ is known to be 1.

c) Assuming the regularity conditions for QMLE are met, derive the asymptotic distribution for $\hat{\beta}$. Note, you do not need to derive this from first principles, you may use QMLE results.

d) Given the distribution of $\hat{\beta}$, find the distribution of $\ln \hat{\beta}$.

e) Instead, given $\hat{\beta}$ from QMLE, describe in steps how you would construct a 95% confidence interval for $\ln \beta$ with bootstrap.

f) Suppose Instead that you asked to construct a 95% confidence interval with the bootstrap for $\ln \beta$ from the least squares regression,

$$y_t - \ln y_{t-1} = \ln \beta + \epsilon_t$$

with $\sigma_\epsilon = 1$ and known. Is the percentile method pivotal? Does the bootstrap provide an asymptotic refinement in this case?

g) For an initial guess for β, say β_1, construct the first step in the BHHH algorithm.

[7] Consider a log-linearized present value model,

$$d_t - p_t = \rho^{-1} E_t \sum_{j=1}^{\infty} \rho^j (r_{t+j} - \Delta d_{t+j}),$$

where Δd_{t+j} dividend growth, $d_t - p_t$ is is the log of the dividend-price ratio, and r_t is the log of the gross return. Using a log-linear approximation, r_t can be approximated as

$$r - t = \Delta d_t - \rho(d_t - p_t) + (d_{t-1} - p_{t-1}) \, .$$

The parameter $0 < \rho < 1$ is a discount factor, which we assume is known, and constants are suppressed for convenience.

a) Suppose d_t has a unit root. For studying this relationship, what would be a sensible specification for a vector time-series model? Explain your modeling choice.

b) Suppose you wanted to test the hypothesis that expected returns are constant. This imposes a collection of nonlinear cross-equation restrictions on your vector time-series model that we can represent as $R(\theta) = 0$. Describe how you would test those restrictions.

c) What is the asymptotic distribution of your test statistic? Explain.